Designing Binary Sequence Sets for MIMO Radar Systems

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***Abstract*—In this paper, we aim at designing a set of binary sequences with good auto- and cross-correlation properties minimizing a weighted sum of Peak Sidelobe Level (PSL) and Integrated Sidelobe Level (ISL) for Multiple Input Multiple Output (MIMO) radar systems. To formulate the problem, we introduce a Pareto-objective of PSL and ISL to measure the aperiodic/periodic correlation sidelobes and establish a multi-objective constrained problem. Then, by using a two-step coordinate descent (CD) framework, we propose an efficient monotonic algorithm based on Fast Fourier Transform (FFT), to directly minimize the objective function. Numerical results show that the proposed algorithm can provide better performance than the state-of-the-art algorithms.**

***Index Terms*—Binary Sequences, Integrated Sidelobe Level (ISL), Multiple Input Multiple Output (MIMO), Peak Sidelobe Level (PSL), Waveform Design.**

I. Introduction

MIMO radar systems usually radiate orthogonal (or incoherent) waveforms by their transmit antennas [1], [2] to allow the matched filters separating them at receive side [3]. If the codes (waveforms) have any non-zero cross-correlation sidelobes, the energy will leak from one waveform to other waveforms in the receiver matched filter [4] and this affects improperly the system performance. Hence, a successful design of orthogonal sequence sets from a family of constant modulus alphabet, with "good" auto- and cross-correlation properties is crucial for MIMO radar systems [4]. Additionally, in order to use a set of sequences in a practical radar system, it requires that the set is chosen from a specific family of constellation, i.e., discrete phase sequences [5].

Let us consider a MIMO radar system with N_T transmit antennas. Each antenna transmits a code vector which is composed of N sub-pulses (intra-pulse coding) and can be written at the m-th transmit antenna as,

$$\boldsymbol{x}_m = [x_m(1), x_m(2), \ldots, x_m(N)]^T \in \mathbb{C}^N, \tag{1}$$

where $x_m(n)$ is the n-th sub-pulse of the transmit code vector $\boldsymbol{x}_m$. Let $\{\boldsymbol{x}_m\}_{m=1}^{N_T}$ be columns of the code matrix $\boldsymbol{X}$, viz. ,

$$\boldsymbol{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{N_T}] \in \mathbb{C}^{N \times N_T}. \tag{2}$$

The aperiodic cross-correlation [6] of $\{x_m(n)\}_{n=1}^N$ and $\{x_l(n)\}_{n=1}^N$ at lag k is defined as,

$$r_{ml}^{\mathcal{AP}}(k) = \sum_{n=1}^{N-k} x_m(n) x_l^*(n+k) = r_{lm}^{\mathcal{AP}*}(-k), \quad m,l = 1,\ldots,N_T, \quad -N+1 \le k \le N-1, \tag{3}$$

when $m = l$, equation (3) becomes the aperiodic auto-correlation of $\{x_m(n)\}_{n=1}^N$. Similarly, we define periodic cross-correlation [7] of $\{x_m(n)\}_{n=1}^N$ and $\{x_l(n)\}_{n=1}^N$ at lag k as,

$$r_{ml}^{\mathcal{P}}(k) = \sum_{n=1}^{N} x_m(n) x_l^*(n+k)_{\mathrm{mod}(N)} = r_{lm}^{\mathcal{P}*}(-k), \quad m,l = 1,\ldots,N_T, \quad -N+1 \le k \le N-1. \tag{4}$$

Again, setting $m = l$, the $r_{ml}^{\mathcal{P}}(k)$ becomes the periodic auto-correlation of $\{x_m(n)\}_{n=1}^N$. For the simplicity, in the sequel we use the notation of $r_{ml}(k)$ to show both aperiodic and periodic correlation functions. Precisely, setting $r_{ml}(k) = r_{ml}^{\mathcal{AP}}(k)$ and $r_{mm}(k) = r_{mm}^{\mathcal{AP}}(k)$ addresses the aperiodic correlation functions, whereas setting $r_{ml}(k) = r_{ml}^{\mathcal{P}}(k)$ and $r_{mm}(k) = r_{mm}^{\mathcal{P}}(k)$ addresses the periodic correlation function.

Two commonly used metrics for the goodness of the correlation function for the code matrix $\boldsymbol{X}$ are the PSL and the ISL which are defined as [8],

$$\mathrm{PSL} = \max\left\{ \max_{m}\left\{ \max_{k\neq 0} |r_{mm}(k)|^2 \right\}, \max_{\substack{m,l\\ m\neq l}} \left\{ \max_{k} |r_{ml}(k)|^2 \right\} \right\}, \tag{5}$$

$$\mathrm{ISL} = \sum_{m=1}^{N_T} \sum_{\substack{k=-N+1\\ k\neq 0}}^{N-1} |r_{mm}(k)|^2 + \sum_{\substack{m,l=1\\ m\neq l}}^{N_T} \sum_{k=-N+1}^{N-1} |r_{ml}(k)|^2 \tag{6}$$

In a MIMO radar system, we interest in achieving a set of sequences with small PSL and low ISL. Small auto-correlation sidelobes, indicates that any sequence in the set is approximately uncorrelated with its own time shifted versions and therefore it avoids masking weak targets within the range sidelobes of a strong target; whereas a good cross-correlation means that any member of the sequences in the set is roughly uncorrelated with any other members at any shift, which leads to better separation of the matched filtered outputs at receive

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side [9]. In a pulse MIMO radar system [10], the aperiodic auto- and cross-correlation functions should have low PSL and small ISL while in a continuous wave MIMO radar system [11], due to the succession in transmission, the good properties must be held on periodic auto- and cross-correlation functions. The application of designing sequences with good cross-correlation properties, can also be found in telecommunication systems. In Code Division Multiple Access (CDMA), sequences with zero periodic out of phase correlation sidelobes are widely used for reliable synchronization [12], [13]. In this respect, the well known families of Frank, Golomb, P1, P2, P3, P4, PX, M-Sequences, Gold, Kasami and etc. have been introduced and used [14]–[22] in different communication systems. Unfortunately, these families are not perfectly matched with the MIMO radar requirements, since they have no constraint on their auto- and cross-correlation functions simultaneously. Therefore, various algorithms have been developed for sequence set design in MIMO radar systems [14], [23]–[26] to be discussed below.

A. Background and Related Works

In classical pulse compression approaches, sequences are designed to have good aperiodic/periodic auto-correlation properties (small PSL and low ISL) [27]–[31]. Some examples are binary sequences such as the Barker codes (known up to length 13) [32], minimum peak sidelobe (MPS) [33] (known up to length 105), M-sequences, Gold or Kasami sequences (defined for the specific lengths of $2^n - 1$ when n is a natural number) [34] and polyphase sequences such as generalized barker codes (known up to length 77) codes [35], Frank, P1, P2, P3 (defined for the lengths that are perfect squares) [15], [36], Golomb, P4, Px, Zadoff-Chu. In order to overcome the mentioned limitations, new researches have provided some sets via optimization-oriented algorithms to design sequences with good correlation properties [4], [14], [19], [21], [31], [37]–[45]. Among them, CAN (cyclic algorithm-new), We-CAN [37], ITROX [41], MWISL, MWISL-Diag [44] deal with minimization of the ISL metric, whereas MM-PSL [44], PMAR, POCA and RPOCA deal with the minimization of a PSL related metric (lp-norm minimization) [45]. Besides, CPM and DPM [31] consider a weighted function of both PSL and ISL in the minimization stage.

In recent years, a large number of researches has been devoted to design waveform sets for MIMO radar systems, to enhance transmit beamforming [46]–[50], improve target detection performance [51]–[54], promote radar spatial resolution [6], [7], [14], [20], [24], [26], [41], [55]–[58] and obtain better target classification/recognition performance [59]–[61]. The constraint sets considered in the design stage usually are the energy, Peak-to-Average-power ratio (PAR), constant modulus and discrete phase. Notice that, the most heavy constraint in the optimization stage is the binary phase which is practically more important.

Designing set of sequences with good correlation properties (see (5),(6)), is an important line of research for MIMO systems. In this respect, the simulated annealing (SA) [20] and cross entropy-based [55] methods are proposed for designing sequences with good auto- and cross-correlation properties. However, due to the high computational complexity, these methods can not be used to efficient design of medium length sequences. Therefore, Multi-CAN [4] algorithm is proposed for fast design of unimodular orthogonal set of sequences, minimizing an almost-equivalent metric of the ISL. This algorithm is developed for design of set of sequences in both cases of good aperiodic and periodic[1] correlation function. Subsequently, extending the CAN algorithm, a fast algorithm named CANARY (CAN complementary) is developed for designing sets of complementary sequences with good correlation properties [26]. Meanwhile, based on the majorization-minimization (MM) technique, the MM-Corr algorithm is proposed [6] to directly minimize the ISL being computationally more effective than Multi-CAN. However, none of the aforementioned methods considers the design problem with the minimization of the PSL imposing binary constraint.

B. Contributions

In this paper we propose a mathematical approach for designing binary (discrete phase) sequences set with application in MIMO radar systems. The proposed method can be used in the other signal processing applications including spread spectrum communications, channel estimation and fast start-up equalization, sonar systems, etc. Precisely, the major contributions of this paper lie in the following three aspects:

1) We have proposed sequence sets that exhibit both low auto- and cross-correlation sidelobes in terms of PSL.
2) We have tackled the binary design problem by minimizing a weighted sum of PSL and ISL for the code matrix $\boldsymbol{X}$.
3) The proposed method has the ability of designing discrete phase sequences with arbitrary constellation sizes.

C. Organization and Notation

The rest of this work is organized as follows. In Section II, the design problem is formulated. In Section III, we develop a two-stage coordinate descent (CD) framework to deal with the problem. In Section IV, the solution to the scalar sub-problem for each iteration of CD is derived. Section V provides several numerical experiments to verify the effectiveness of the proposed algorithm. Finally, Section VI concludes the paper.

The following notation is adopted in the paper. Bold lowercase letters for vectors and bold uppercase letters for matrices. The transpose, the conjugate, and the conjugate transpose operators are denoted by the symbols $(\cdot)^T$, $(\cdot)^*$ and $(\cdot)^H$ respectively. The letter $\jmath$ represents the imaginary unit (i.e., $\jmath = \sqrt{-1}$). For any $x \in \mathbb{R}$, $|x|$ and $\arg(x)$ represent the modulus and the argument of x, respectively. The n-th element of the vector $\boldsymbol{x}$ is denoted by $x(n)$. The abbreviation "s.t." stands for "subject to".

II. Problem Formulation

In this section, we cast the code design problem to obtain a set of sequences with small out-of-phase auto-correlation

[1]In this case the algorithm is named Multi Pe-CAN.

and low cross-correlation sidelobes. Note that, the in-phase lag (i.e., $k=0$) of both auto-correlation functions (periodic and aperiodic) represents the energy component of the sequence whereas the out-of-phase (i.e., $k \neq 0$) represent the sidelobes. Let define the following metrics for the code matrix $\boldsymbol{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{N_T}]$,

$$\widetilde{f}_1(\boldsymbol{X}) = \max_{m} \left\{ \max_{k \neq 0} |r_{mm}(k)|^2 \right\}, \tag{7}$$

$$\widetilde{f}_2(\boldsymbol{X}) = \max_{\substack{m,l \\ m \neq l}} \left\{ \max_{k} |r_{ml}(k)|^2 \right\}, \tag{8}$$

$$\widetilde{f}_3(\boldsymbol{X}) = \sum_{m=1}^{N_T} \sum_{\substack{k=-N+1 \\ k \neq 0}}^{N-1} |r_{mm}(k)|^2, \tag{9}$$

and

$$\widetilde{f}_4(\boldsymbol{X}) = \sum_{\substack{m,l=1 \\ m \neq l}}^{N_T} \sum_{k=-N+1}^{N-1} |r_{ml}(k)|^2, \tag{10}$$

where $\widetilde{f}_1(\boldsymbol{X})$ is the maximum auto-correlation value of all N_T transmitting codes whereas $\widetilde{f}_2(\boldsymbol{X})$ is the maximum cross-correlation value between all different N_T sequences $\{\boldsymbol{x}_m\}_{m=1}^{N_T}$. Correspondingly, $\widetilde{f}_3(\boldsymbol{X})$ is the summation of the auto-correlation sidelobes of all different N_T sequences whereas $\widetilde{f}_4(\boldsymbol{X})$ is the summation of the cross-correlation between all different N_T codes. We aim to design a good set of sequences $\boldsymbol{X}^\star$ by minimizing all the objective functions $\{\widetilde{f}_i(\boldsymbol{X})\}_{i=1}^4$ simultaneously. Since the design problem is constrained to the families of constant modulus discrete phase sequences, the n-th sub-pulse at m-th transmit antenna can be written as,

$$x_m(n) = e^{\jmath \phi_m(n)}, \;\; m = 1, \ldots, N_T \text{ and } n = 1, \ldots, N \tag{11}$$

with $\phi_m(n)$ being the phase of the n-th subpulse of the transmit code vector $\boldsymbol{x}_m$. The phase $\phi_m(n)$ can only be selected from the following set:

$$\phi_m(n) \in \left\{ 0, \frac{2\pi}{L}, \ldots, \frac{(L-1)2\pi}{L} \right\} \triangleq \phi_L \tag{12}$$

where L is the number of distinct phase values. Let $\bar{\omega} = e^{\jmath \frac{2\pi}{L}}$ and $\Psi_L = \{1, \bar{\omega}, \ldots, \bar{\omega}^{L-1}\}$, the feasibility region for discrete phase code design problem can be shown as

$$\Omega_L = \{\boldsymbol{x}_m | x_m(n) \in \Psi_L, n = 1, \ldots, N\}. \tag{13}$$

Therefore, the optimization problem can be cast as,

$$P_{\boldsymbol{X}} = \begin{cases} \min\limits_{\boldsymbol{X}} & \widetilde{f}_1(\boldsymbol{X}), \widetilde{f}_2(\boldsymbol{X}), \widetilde{f}_3(\boldsymbol{X}), \widetilde{f}_4(\boldsymbol{X}) \\ \text{s.t.} & \boldsymbol{x}_{m,l} \in \Omega_L, \;\; m,l = 1, \ldots, N_T \end{cases} \tag{14}$$

which is a multi-objective non-convex optimization problem. In a multi-objective optimization problem, usually a feasible solution that minimizes all the objective functions simultaneously does not exist [62]. A viable means to handle these type of problems, is to use the scalarization technique[2] which exploits as objective a specific weighted sum between the objective functions. Specifically, we define $\widetilde{f}_w(\boldsymbol{X})$, parameterized in the weighting coefficient $w \in [0,1]$ as,

$$\widetilde{f}_w(\boldsymbol{X}) = w \max \left\{ \widetilde{f}_1(\boldsymbol{X}), \widetilde{f}_2(\boldsymbol{X}) \right\} + (1-w) \left\{ \widetilde{f}_3(\boldsymbol{X}) + \widetilde{f}_4(\boldsymbol{X}) \right\}, \tag{15}$$

where the first term indicates the PSL defined by (5) while the second term shows the ISL stated in (6). The scalarization leads to the following design problem,

$$P_{\boldsymbol{X}}^w = \begin{cases} \min\limits_{\boldsymbol{X}} & \widetilde{f}_w(\boldsymbol{X}) \\ \text{s.t.} & \boldsymbol{x}_{m,l} \in \Omega_L, \;\; m,l = 1, \ldots, N_T \end{cases} \tag{16}$$

In the next sections, we devise an efficient algorithm to deal with the design Problem $P_{\boldsymbol{X}}^w$.

III. The Proposed Method

In this section, we devise an iterative derivative-free optimization method based on the Coordinate Descent (CD) minimization procedure [64] (also known as alternate optimization [65]) to sequentially optimize the objective function $P_{\boldsymbol{X}}^w$ over one variable keeping fixed the others. The general idea to tackle $P_{\boldsymbol{X}}^w$ using the CD algorithm is shown below:

1) Pick coordinate t from $1, 2, \ldots, N_T$.
2) Set $\boldsymbol{x}_t^{(i+1)} = \arg\min\limits_{\boldsymbol{x}_t} \widetilde{f}_w(\boldsymbol{x}_t, \boldsymbol{X}_{-t}^{(i)})$.

where $\boldsymbol{X}_{-t}^{(i)}$ represent all other coordinates at iteration $(i+1)$ which are keeping fixed. Also, to obtain the optimal code entry $x_t(d)$, we use CD algorithm as below:

1) Pick coordinate d from $1, 2, \ldots, N$.
2) Set $x_t^{(h+1)}(d) = \arg\min\limits_{x_t(d)} f_w(x_t(d), \boldsymbol{x}_{t,-d}^{(h)})$.

where $\boldsymbol{x}_{t,-d}^{(h)}$ represent all other coordinates of the code vector $\boldsymbol{x}_t$ at iteration $(h+1)$ which are keeping fixed. Thus, the CD framework solves optimization problems by successively performing minimization along coordinate directions or hyperplanes. Indeed, in a nutshell to handle the Problem $P_{\boldsymbol{X}}^w$ we resort into the following subproblems:

- **Outer loop;** Designing a code vector $\boldsymbol{x}_t$ keeping fixed the other code vectors.
- **Inner loop;** Optimizing each scaler variable $x_t(d)$ of $\boldsymbol{x}_t$, keeping fixed the other entries of the code $\boldsymbol{x}_t$.

Therefore, by solving a sequence of simpler optimization problems, each subproblem will have a lower dimension in the minimization procedure, and thus can typically be solved easier than the original problem. Accordingly, we perform the optimization procedure cyclically in an outer and inner loop as specified below.

A. Outer loop

In the outer loop, we select the code vector $\boldsymbol{x}_t$, to be optimized while keeping the others fixed. The optimization Problem $P_{\boldsymbol{X}}^w$ at iteration $(i+1)$ boils down to,

$$P_{t,\boldsymbol{X}^{(i)}}^w = \begin{cases} \min\limits_{\boldsymbol{x}_t} & \widetilde{f}_w(\boldsymbol{x}_t, \boldsymbol{X}_{-t}^{(i)}) \\ \text{s.t.} & \boldsymbol{x}_t \in \Omega_L, \end{cases} \tag{17}$$

[2] Scalarizing a multi-objective problem involves the solution of conventional optimization problems whose objective function is a specific convex combination of the original figures of merits [63].

where

$$\boldsymbol{X}_{-t}^{(i)} = \left[\boldsymbol{x}_1^{(i)}, \boldsymbol{x}_2^{(i)}, \ldots, \boldsymbol{x}_{t-1}^{(i)}, \boldsymbol{x}_{t+1}^{(i)}, \ldots, \boldsymbol{x}_{N_T}^{(i)}\right] \in \mathbb{C}^{N \times N_T - 1} \tag{18}$$

refers to the remaining sequences in $\boldsymbol{X}$ other than $\boldsymbol{x}_t$, and

$$\widetilde{f}_w(\boldsymbol{x}_t, \boldsymbol{X}_{-t}^{(i)}) = \widetilde{f}_w\left(\boldsymbol{x}_1^{(i)}, \boldsymbol{x}_2^{(i)}, \ldots, \boldsymbol{x}_{t-1}^{(i)}, \boldsymbol{x}_t, \boldsymbol{x}_{t+1}^{(i)}, \ldots, \boldsymbol{x}_{N_T}^{(i)}\right). \tag{19}$$

Thus, denoting by $\boldsymbol{X}_t^{\star(i+1)}$ the optimal solution to $P^w_{t,\boldsymbol{X}^{(i)}}$, the optimized code matrix at iteration $(i+1)$ becomes,

$$\boldsymbol{X}_t^{\star(i+1)} = \left[\boldsymbol{x}_1^{(i)}, \boldsymbol{x}_2^{(i)}, \ldots, \boldsymbol{x}_{t-1}^{(i)}, \boldsymbol{x}_t^{\star}, \boldsymbol{x}_{t+1}^{(i)}, \ldots, \boldsymbol{x}_{N_T}^{(i)}\right]. \tag{20}$$

As a result, starting from an initial code matrix $\boldsymbol{X}^{(0)}$, the code matrices $\boldsymbol{X}^{(1)}$, $\boldsymbol{X}^{(2)}$, $\boldsymbol{X}^{(3)}$, ... are obtained iteratively[3].

B. Inner loop

In the inner loop, we aim to optimize the code entries of the selected code vector $\boldsymbol{x}_t$. Again, using the CD algorithm, in the inner loop and in iteration $(h+1)$, we choose the scaler $x_t(d)$ as the variable to be optimized and put the remaining code entries in the vector $\boldsymbol{x}_{t,-d}^{(h)} \in \mathbb{C}^{N-1}$ defined as,

$$\boldsymbol{x}_{t,-d}^{(h)} = [x_t^{(h)}(1), \ldots, x_t^{(h)}(d-1), x_t^{(h)}(d+1), \ldots, x_t^{(h)}(N)]^T. \tag{21}$$

Consequently, the optimization problem at the inner loop in iteration $(h+1)$ is,

$$P^w_{d,\boldsymbol{x}_t^{(h)}} \begin{cases} \min\limits_{x_t(d)} & f_w\left(x_t(d); \boldsymbol{x}_{t,-d}^{(h)}\right) \\ \text{s.t.} & x_t(d) \in \Omega_L \end{cases} \tag{22}$$

where

$$f_w\left(x_t(d); \boldsymbol{x}_{t,-d}^{(h)}\right) = f_w\Big(x_t^{(h)}(1), x_t^{(h)}(2), \ldots, x_t^{(h)}(d-1), x_t(d), x_t^{(h)}(d+1), \ldots, x_t^{(h)}(N-1), x_t^{(h)}(N)\Big). \tag{23}$$

Thus, denoting $x_t^{\star}(d)$ as the optimal solution to (22), the optimized code vector for the t-th transmit antenna at iteration $(h+1)$ will be,

$$\boldsymbol{x}_t^{(h+1)} = [x_t^{(h)}(1), x_t^{(h)}(2), \ldots, x_t^{\star}(d), \ldots, x_t^{(h)}(N)]^T.$$

As a result, starting from an initial code $\boldsymbol{x}_t^{(0)}$, a sequence $\boldsymbol{x}_t^{(1)}, \boldsymbol{x}_t^{(2)}, \boldsymbol{x}_t^{(3)}, \ldots$ are obtained iteratively.

Remark 1. *In order to tackle the optimization Problem* (22)*, we define,*

$$f_1(\boldsymbol{x}_t) = \max_{k \neq 0} |r_{tt}(k)|^2, \tag{24}$$

[3]The super scripts (i) and $(i+1)$ for $\boldsymbol{x}_t$ and $\boldsymbol{x}_t^{\star}$ are implicit due to the simplicity.

Algorithm 1 BiST algorithm for MIMO radar systems

Input: Initial code matrix $\boldsymbol{X}^{(0)} \in \mathbb{C}^{N \times N_T}$, $w \in [0, 1]$;
Output: Optimal sequence set $\boldsymbol{X}^{\star}$;

1) **Initialization**.
   - Compute the initial objective value $\widetilde{f}_w(\boldsymbol{x}_1^{(0)}, \boldsymbol{x}_2^{(0)}, \ldots, \boldsymbol{x}_{N_T}^{(0)})$ using equation (19);
   - Set $t := 1$ and $i := 0$;
2) **Improvement**.
   - Solve $P^w_{t,\boldsymbol{X}^{(i)}}$ and obtain $\boldsymbol{x}_t^{\star}$ by doing the following steps:

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     a) **Initialization**.
        - Compute the initial objective value $f_w(x_t^{(0)}(1), x_t^{(0)}(2), \ldots, x_t^{(0)}(N))$ using equation (23);
        - Set $d := 1$ and $h := 0$;

     b) **Improvement**.
        - Solve $P^w_{d,\boldsymbol{x}_t^{(h)}}$ and obtain $x_t^{\star}(d)$;
        - Set $h := h + 1$ and $\boldsymbol{x}_t^{(h)} = \left[x_t^{(h-1)}(1), \ldots, x_t^{\star}(d), \ldots, x_t^{(h-1)}(N)\right]^T$;

     c) **Stopping Criterion**.
        - If $|f_w(\boldsymbol{x}_t^{(h)}) - f_w(\boldsymbol{x}_t^{(h-1)})| < \epsilon$, stop. Otherwise, update d, i.e., if $d < N$, $d = d + 1$, otherwise $d = 1$, and go to the step b;

     d) **Output**.
        - Set $\boldsymbol{x}_t^{\star} = \boldsymbol{x}_t^{(h)}$.

   - Set $i := i + 1$ and $\boldsymbol{X}_t^{(i)} = \left[\boldsymbol{x}_1^{(i-1)}, \boldsymbol{x}_2^{(i-1)}, \ldots, \boldsymbol{x}_t^{\star}, \ldots, \boldsymbol{x}_{N_T}^{(i-1)}\right]$;
3) **Stopping Criterion**.
   - If $|\widetilde{f}_w(\boldsymbol{X}_t^{(i)}) - \widetilde{f}_w(\boldsymbol{X}_t^{(i-1)})| < \epsilon$, stop. Otherwise, update t, i.e., if $t < N_T$, $t = t+1$, otherwise $t = 1$, and go to the step 2;
4) **Output**.
   - Set $\boldsymbol{X}_t^{\star} = \boldsymbol{X}_t^{(i)}$.

$$f_2(\boldsymbol{x}_t) = \max_{\substack{l \\ l \neq t}} \left\{\max_k |r_{tl}(k)|^2\right\}, \tag{25}$$

$$f_3(\boldsymbol{x}_t) = \sum_{\substack{k=-N+1 \\ k \neq 0}}^{N-1} |r_{tt}(k)|^2, \tag{26}$$

and

$$f_4(\boldsymbol{x}_t) = \sum_{\substack{l=1 \\ l \neq t}}^{N_T} \sum_{k=-N+1}^{N-1} |r_{tl}(k)|^2, \tag{27}$$

where $f_1(\boldsymbol{x}_t)$ is the maximum auto-correlation sidelobe of t-th transmitting waveform whereas $f_2(\boldsymbol{x}_t)$ is the maximum cross-correlation value between the t-th code vector and the remained code vectors in the sequence set. Similarly, $f_3(\boldsymbol{x}_t)$ is the summation of the sidelobes of t-th waveform whereas

$f_4(\boldsymbol{x}_t)$ is the summation of cross-correlation between t-th waveform and the remained sequences. Consequently,

$$f_w\left(x_t(d);\boldsymbol{x}_{t,-d}^{(h)}\right) = w\max\{f_1(\boldsymbol{x}_t), f_2(\boldsymbol{x}_t)\} \tag{28}$$
$$+ (1-w)\{f_3(\boldsymbol{x}_t)+f_4(\boldsymbol{x}_t)\}. \blacksquare \tag{29}$$

Finally, to obtain the optimal sequence set $\boldsymbol{X}^\star$, we solve Problem $P^w_{d,\boldsymbol{x}_t^{(h)}}$ sequentially and find the optimal code vector $\boldsymbol{x}_t^\star$ at the inner loop and tackle Problem $P^w_{t,\boldsymbol{X}^{(i)}}$ to obtain the optimal set of sequences $\boldsymbol{X}^\star = [\boldsymbol{x}_1^\star, \boldsymbol{x}_2^\star, \ldots, \boldsymbol{x}_{N_T}^\star]$ at the outer loop. A summary of the proposed approach to obtain $\boldsymbol{X}^\star$ can be found in **Algorithm 1**. Since the proposed approach can effectively design binary set of sequences, we name it **Binary Sequences seT (BiST)**[4] method.

To proceed further, in the following we make explicit the functional dependence of the objective function in $P^w_{d,\boldsymbol{x}_t^{(h)}}$, i. e., $f_w\left(x_t(d);\boldsymbol{x}_{t,-d}^{(h)}\right)$ over the optimization variable $x_t(d)$.

1) **Aperiodic Correlation Function:** Let us start with the aperiodic auto-correlation function of the t-th selected transmit code $\boldsymbol{x}_t$. As to the optimization variable $x_t(d)$ in (22), it is verified that,

$$r_{tt}^{\mathcal{AP}}(k) = a_{dkt}^{\mathcal{AP}}x_t(d) + b_{dkt}^{\mathcal{AP}}x_t^*(d) + c_{dkt}^{\mathcal{AP}}, \quad k=-N+1,\ldots,N-1 \tag{30}$$

where[5],

$$a_{dkt}^{\mathcal{AP}} \triangleq x_t^{*^{(h)}}(d+k)\mathbf{I}_A(d+k), \tag{31}$$
$$b_{dkt}^{\mathcal{AP}} \triangleq x_t^{(h)}(d-k)\mathbf{I}_A(d-k), \tag{32}$$

and

$$c_{dkt}^{\mathcal{AP}} \triangleq \sum_{n=1,n\neq\{d,d-k\}}^{N-k} x_t^{(h)}(n)x_t^{*(h)}(n+k)\mathbf{I}_A(k+1) + \sum_{n=-k+1,n\neq\{d,d-k\}}^{N} x_t^{(h)}(n)x_t^{*(h)}(n+k)\mathbf{I}_B(k) \tag{33}$$

with $\mathbf{I}_A(k)$ and $\mathbf{I}_B(k)$ being the indicator functions of sets $A=\{1,2,\ldots,N\}$ and $B=\{-1,-2,\ldots,-N+1\}$ respectively, i.e., $\mathbf{I}_A(v)=1$ if $v\in A$, otherwise $\mathbf{I}_A(v)=0$.

Similarly, the cross-correlation function $r_{tl}^{\mathcal{AP}}(k)$ with explicit dependence on $x_t(d)$ becomes,

$$r_{tl}^{\mathcal{AP}}(k) = a_{dkl}^{\mathcal{AP}}x_t(d) + c_{dkl}^{\mathcal{AP}},\ k=-N+1,\ldots,N-1. \tag{34}$$

with

$$a_{dkl}^{\mathcal{AP}} \triangleq x_l^{*(h)}(d+k)\mathbf{I}_A(d+k),\ k=-N+1,\ldots,N-1. \tag{35}$$

and

$$c_{dkl}^{\mathcal{AP}} \triangleq \sum_{n=1,n\neq d}^{N-k} x_t^{(h)}(n)x_l^{*(h)}(n+k)\mathbf{I}_A(k+1) + \sum_{n=-k+1,n\neq d}^{N} x_t^{(h)}(n)x_l^{*(h)}(n+k)\mathbf{I}_B(k), \tag{36}$$

[4]BiST means the number "20" in Persian language and also refers to something that is perfect.

[5]For notational simplicity, the dependence of auto/cross correlation on h is implicit.

2) **Periodic Correlation Function:** Next, we provide the functional dependence of the periodic correlation function on the variable $x_m(d)$. As to the auto-correlation function, we can write,

$$r_{tt}^{\mathcal{P}}(k) = a_{dkt}^{\mathcal{P}}x_t(d) + b_{dkt}^{\mathcal{P}}x_t^*(d) + c_{dkt}^{\mathcal{P}}, \quad k=-N+1,\ldots,N-1. \tag{37}$$

where

$$a_{dkt}^{\mathcal{P}} \triangleq x_t^{*(h)}(d+k)\mathbf{I}_A(d+k) + x_t^{*(h)}(d+k-N)\mathbf{I}_A(d+k-N) \tag{38}$$

and

$$b_{dkt}^{\mathcal{P}} \triangleq x_t^{(h)}(d-k)\mathbf{I}_A(d-k) + x_t^{(h)}(d-k+N)\mathbf{I}_A(d-k+N) \tag{39}$$

with

$$c_{dkt}^{\mathcal{P}} \triangleq \sum_{n=1,n\neq\{d,d-k\}}^{N-k} x_t^{(h)}(n)x_t^{*(h)}(n+k) \tag{40}$$
$$+ \sum_{n=N-k+1,n\neq\{d,d-k+N\}}^{N} x_t^{(h)}(n)x_t^{*(h)}(n+k-N). \tag{41}$$

Also, for the periodic cross-correlation function we have,

$$r_{tl}^{\mathcal{P}}(k) = a_{dkl}^{\mathcal{P}}x_t(d) + c_{dkl}^{\mathcal{P}},\ k=-N+1,\ldots,N-1. \tag{42}$$

where

$$a_{dkl}^{\mathcal{P}} \triangleq x_l^{*(h)}(d+k)\mathbf{I}_A(d+k) + x_l^{*(h)}(d+k+N)\mathbf{I}_A(d+k+N) + x_l^{*(h)}(d+k-N)\mathbf{I}_A(d+k-N), \tag{43}$$

and

$$c_{dkl}^{\mathcal{P}} \triangleq \sum_{n=1,n\neq d}^{N}\Big[x_t^{(h)}(n)x_l^{*(h)}(n+k)\mathbf{I}_A(n+k) + x_t^{(h)}(n)x_l^{*(h)}(n+k+N)\mathbf{I}_A(n+k+N) + x_t^{(h)}(n)x_l^{*(h)}(n+k-N)\mathbf{I}_A(n+k-N)\Big]. \tag{44}$$

In the next section, we devise a method for dealing with (22).

IV. The Code Entry Design

In this section, we devise an efficient algorithm to find the optimal solution of the following non-convex, constrained problem,

$$P^w_{d,\boldsymbol{x}_t^{(h)}}\begin{cases}\min\limits_{x_t(d)} & f_w\left(x_t(d);\boldsymbol{x}_{t,-d}^{(h)}\right)\\ \text{s.t.} & x_t(d)\in\Omega_L\end{cases} \tag{45}$$

According to the derived dependency of the correlation functions to the complex variable $x_t(d)$ in the previous section, we observe,

$$f_w\left(x_t(d);\boldsymbol{x}_{t,-d}^{(h)}\right) = w\max\left\{\max_{k\neq 0}\left|a_{dkt}x_t(d)+b_{dkt}x_t^*(d)+c_{dkt}\right|^2, \max_{\substack{l\\ l\neq t}}\left[\max_k\left|a_{dkl}x_t(d)+c_{dkl}\right|^2\right]\right\} + (1-w)\left\{\sum_{\substack{k=-N+1\\ k\neq 0}}^{N-1}\left|a_{dkt}x_t(d)+b_{dkt}x_t^*(d)+c_{dkt}\right|^2 + \sum_{\substack{l=1\\ l\neq t}}^{N_T}\sum_{k=-N+1}^{N-1}\left|a_{dkl}x_t(d)+c_{dkl}\right|^2\right\}. \quad (46)$$

Notice that in terms of the phase $\phi_t(d) = \arg(x_t(d))$, the optimization Problem $P^w_{d,\boldsymbol{x}_t^{(h)}}$ can be expressed as,

$$P^w_{d,\phi_t^{(h)}}\begin{cases}\min\limits_{\phi_t(d)} & f_w\left(\phi_t(d)\right)\\ s.t. & \phi_t(d)\in\boldsymbol{\phi}_L\end{cases} \quad (47)$$

with ϕ_L being defined in (12). In the sequel, exploiting Discrete Fourier Transform (DFT), we find the optimal solution $\phi_t^\star(d)$ for the constrained non-convex Problem $P^w_{d,\phi_t^{(h)}}$ assuming two following cases:

A. General Discrete Phase

For $L \geq 3$ and according to the equations (30) and (37) in the optimization Problem $P^w_{d,\phi_t^{(h)}}$, the squared modulus of the auto-correlation functions in correspondence of the phase variable $\phi_t(d)$ can be written as,

$$|r_{tt,\phi_t(d)}(k)|^2 = |a_{dkt}e^{\jmath\phi_t(d)} + b_{dkt}e^{-\jmath\phi_t(d)} + c_{dkt}|^2,$$

correspondingly, according to the equations (34) and (42) for the cross-correlation functions it can be verified that,

$$|r_{tl,\phi_t(d)}(k)|^2 = |a_{dkl}e^{\jmath\phi_t(d)} + c_{dkl}|^2.$$

The following lemma provides a key result to tackle Problem $P^w_{d,\phi_t^{(h)}}$.

Lemma 1 *Let* $\phi_t(q) = \frac{2\pi(q-1)}{L}, \quad q = 1,\ldots,L$,

$$\boldsymbol{\nu}_{dkz} = \left[|r_{tz,\phi_t(1)}(k)|^2, |r_{tz,\phi_t(2)}(k)|^2,\ldots, |r_{tz,\phi_t(L)}(k)|^2\right]^T, \quad (48)$$

where z stands for either t or l and $\boldsymbol{\nu}_{dkz}\in\mathcal{R}^L$.
For $L\geq 3$, also define[6]

$$\boldsymbol{\zeta}_{dkz} = [a_{dkz}, c_{dkz}, b_{dkz}, \mathbf{0}_{1\times(L-3)}]^T \in \mathcal{R}^L.$$

Then,

$$\boldsymbol{\nu}_{dkz} = |DFT(\boldsymbol{\zeta}_{dkz})|^2, \quad (49)$$

with $DFT(\boldsymbol{\zeta}_{dkz})$ is the L-points DFT of the vectors $\boldsymbol{\zeta}_{dkz}$. Notice that the square modulus is performed element wise.

Proof. See Appendix A. □

Inspiring from **Lemma 1**, we define the matrix $\boldsymbol{U}_z \in \mathcal{R}^{(2N-1)\times L}$ whose k-th row is $\boldsymbol{\nu}_{dkz}^T$. Notice that, the matrix $\boldsymbol{U}_t$ contains all possible auto-correlation sidelobes[7] of different lags (i. e., k), whereas all the cross-correlation values for different possible lags are written in $\boldsymbol{U}_l$. Let $\boldsymbol{u}_z^p \in \mathcal{R}^L$ and $\boldsymbol{u}_z^s\in\mathcal{R}^L$ be the vectors containing the maximum values and the summation of each columns of the matrix $\boldsymbol{U}_z$, respectively. We can write,

$$\boldsymbol{\omega}_t(d) = w\max\left\{\boldsymbol{u}_t^p, \max_{\substack{l\\ l\neq t}}\boldsymbol{u}_l^p\right\} + (1-w)\left\{\boldsymbol{u}_t^s + \sum_{\substack{l=1\\ l\neq t}}^{N_T}\boldsymbol{u}_l^s\right\}, \quad (50)$$

where $\boldsymbol{\omega}_t(d)\in\mathcal{R}^L$ and the operation of maximum between two vectors is defined element wise. Then, the optimal solution to $P^w_{d,\phi_t^{(h)}}$ is given by

$$\phi_t^\star(d) = \frac{2\pi(q^\star-1)}{L}, \quad (51)$$

where

$$q^\star = \arg\min_{q=1,\ldots,L}\left\{\boldsymbol{\omega}_t(d)\right\}, \quad (52)$$

Hence, based on Lemma 1 and (51), the optimal phase code entry can be efficiently computed as $x_t^\star(d) = e^{\jmath\phi_t^\star(d)}$ using DFT.

B. Binary Sequence Design ($L = 2$)

In the case of designing binary sequences, observe that $x_t(d)\in\{-1,+1\}$ is a real variable, and the aperiodic auto-correlation function in the equation (30) will have a slight modification as,

$$r_{tt}^{\mathcal{AP}}(k) = a_{dkt}^{\mathcal{AP}}x_t(d) + c_{dkt}^{\mathcal{AP}}, k=-N+1,\ldots,N-1.$$

where

$$a_{dkt}^{\mathcal{AP}} \triangleq x_t^{(h)}(d+k)\mathbf{I}_A(d+k) + x_t^{(h)}(d-k)\mathbf{I}_A(d-k)$$

and

$$c_{dkt}^{\mathcal{AP}} \triangleq \sum_{n=1,n\neq\{d,d-k\}}^{N-k} x_t^{(h)}(n)x_t^{(h)}(n+k)\mathbf{I}_A(k+1) + \sum_{n=-k+1,n\neq\{d,d-k\}}^{N} x_t^{(h)}(n)x_t^{(h)}(n+k)\mathbf{I}_B(k)$$

with $\phi_t(d)\in\{0,\pi\}$. Also, according to (37) in the case of periodic auto-correlation function, we can write,

$$r_{tt}^{\mathcal{P}}(k) = a_{dkt}^{\mathcal{P}}x_t(d) + c_{dkt}^{\mathcal{P}},\ k=-N+1,\ldots,N-1.$$

where

$$a_{dkt}^{\mathcal{P}} \triangleq x_t^{(h)}(d+k)\mathbf{I}_A(d{+}k) {+} x_t^{(h)}(d+k-N)\mathbf{I}_A(d{+}k{-}N) + x_t^{(h)}(d-k)\mathbf{I}_A(d-k) + x_t^{(h)}(d-k+N)\mathbf{I}_A(d-k+N)$$

[6]Notice that a_{dkt}, b_{dkt} and c_{dkt} are zero when $k=0$.

[7]Notice that all the values of $\boldsymbol{\nu}_{dkt}$ are zero when $k=0$.

with

$$c_{dkt}^{\mathcal{P}} \triangleq \sum_{n=1, n\neq\{d,d-k\}}^{N-k} x_t^{(h)}(n)x_t^{(h)}(n+k) + \sum_{n=N-k+1, n\neq\{d,d-k+N\}}^{N} x_t^{(h)}(n)x_t^{(h)}(n+k-N).$$

and accordingly $\boldsymbol{\zeta}_{dkt}$ will be defined as,

$$\boldsymbol{\zeta}_{dkt} = [a_{dkt}, c_{dkt}]^T \in \mathcal{R}^2 \tag{53}$$

for both cases of aperiodic and periodic auto-correlation function. The rest of the procedure (i. e., calculation of $\boldsymbol{\nu}_{dkz}$, $\boldsymbol{U}_h$, etc.) is the same as the previous part.

Finally, we provide **Algorithm 2** to precisely show the steps of the proposed approach for solving Problem $P^w_{d,\phi_t^{(h)}}$.

Algorithm 2 Discrete Phase Code Entry Optimization

Input: Initial code vector $\boldsymbol{x}_t^{(h)}$, code entry d, w, and L;
Output: Optimal solution $x_t^\star(d)$;

1) Set $\forall k \in \{-N+1, \ldots, N-1\}$
   - Set
   $$\boldsymbol{\zeta}_{dkt} = \begin{cases} [a_{dkt}, c_{dkt}, b_{dkt}, \mathbf{0}_{1\times(L-3)}]^T & \text{L} \geq 3 \\ [a_{dkt}, c_{dkt}]^T & \text{L = 2} \end{cases}$$
   - Set $\boldsymbol{\zeta}_{dkl} = [a_{dkl}, c_{dkl}, \mathbf{0}_{1\times(L-2)}]^T$;
   - Set $\boldsymbol{\nu}_{dkt} = |\text{FFT}(\boldsymbol{\zeta}_{dkt})|^2$ and $\boldsymbol{\nu}_{dkl} = |\text{FFT}(\boldsymbol{\zeta}_{dkl})|^2$;
2) Define $\boldsymbol{U}_z$, obtain $\boldsymbol{u}_z^p$, $\boldsymbol{u}_z^s$ and calculate
$$\boldsymbol{\omega}_t(d) = w \max\left\{\boldsymbol{u}_t^p, \max_{\substack{l \\ l\neq t}} \boldsymbol{u}_l^p\right\} + (1-w)\left\{\boldsymbol{u}_t^s + \sum_{\substack{l=1 \\ l\neq t}}^{N_T} \boldsymbol{u}_l^s\right\};$$
3) Find the index $q^\star$ where $\boldsymbol{\omega}_t(d)$ is minimum;
4) Set $x_t^\star(d) = e^{\jmath\phi_t^\star(d)}$ with $\phi_t^\star(d) = \frac{2\pi(q^\star-1)}{L}$.

Remark 2. **Algorithm 2** *needs the evaluation of* $(2N-1)$ *different L-points DFTs for auto-correlation functions and* $N_T(2N-1)$ *different L-points DFTs for the cross-correlation function. Each of them can be efficiently computed via a Fast Fourier Transform (FFT) [66]. Therefore the computational complexity order for each waveform* $\boldsymbol{x}_t$ *is* $\mathcal{O}(N_T N L \log_2 L)$ *(at the inner loop) and the overall computational complexity of* **Algorithm 1** *will be* $\mathcal{O}(N_T^2 N L \log_2 L)$.

V. Performance Analysis

We consider a MIMO radar system with $N_T = 4$ transmit antennas, and distinct phase numbers of $L = 2, 4$. We also set the stopping criteria $|\widetilde{f}_w(\boldsymbol{X}^{(i)}) - \widetilde{f}_w(\boldsymbol{X}^{(i-1)})| \leq 10^{-5}$ to terminate the whole procedure while the stopping criteria in design of t-th transmit waveform is $|f_w(\boldsymbol{x}_t^{(h)}) - f_w(\boldsymbol{x}_t^{(h-1)})| \leq 10^{-5}$. As the benchmark for sequences set design with good aperiodic and periodic correlation functions, we use Multi-CAN and Multi-PeCAN algorithms[8], respectively [8]. Notice that, we have included a set of m-sequences whenever the constellation size is $L = 2$. Correspondingly, we have chosen the appropriate code lengths where the m-sequences are defined (i. e., $2^n - 1$, $n = 1, 2, \ldots$) in the simulations.

A. PSL Minimization

We set $w = 1$ in (15) to perform PSL minimization. In the first step and in Fig. 1, we assess the convergence behavior of the BiST algorithm in line with the outer loop iterations. For the both cases of aperiodic (Fig. 1a) and periodic (Fig. 1b) correlation functions, the PSL values obtained via BiST algorithm are illustrated. Precisely, we start from a set of random sequences with code length $N = 63$, settled down in an initialization matrix $\boldsymbol{X}^{(0)}$, and obtain the PSL of the optimal set of sequences $\boldsymbol{X}^\star$.

In Fig. 2, we plot the obtained PSL[9] values via BiST when the code length is $N = \{31, 63, 127, 255, 511\}$. As the benchmark, PSL values of a set of sequences obtained trough Multi-CAN algorithm are illustrated in this figure. Notice that, the random starting set of sequences for both BiST and Multi-CAN algorithms, is the same[10]. Besides, the PSL values of a set of m-sequences are also plotted in Fig. 2a where the constellation size is $L = 2$, whereas the PSL values corresponding to the constellation size $L = 4$, are plotted in Fig. 2b.

In Fig. 3, we irritate the same simulation as for the Fig. 2, but considering periodic correlation function. Consequently, we use Multi-PeCAN (instead of Multi-CAN) algorithm as the benchmark. Notice that, in both cases of aperiodic and periodic correlation functions, the set of obtained sequences via the BiST algorithm, has a significant gain over its counterparts for all different lengths. As we can see from the figures 2 and 3, the set of random sequences achieve lower PSL values than the set of m-sequences. This implies that in a MIMO radar system, the use of m-sequences is not recommended (due to high values of the cross-correlation lags).

Finally, Fig. 4 illustrates auto-correlation (Fig. 4a) and cross-correlation (Fig. 4b) levels of a typical set of sequences obtained through BiST where $N = 63$ and $L = 2$. This figure, provides a visual understanding of the goodness of the proposed algorithm.

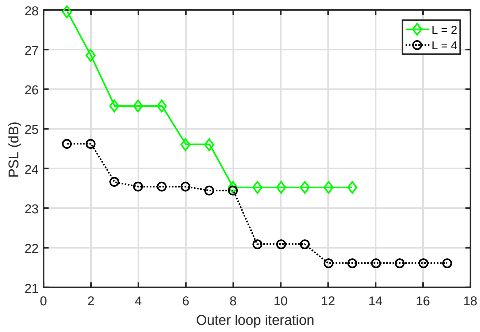
B. ISL Minimization

We set $w = 0$ in (15) to resort to the problem of ISL minimization. Again, we provide the convergence behavior of the BiST algorithm in the first step. We set $N = 63$ and start from a set of random sequences to plot Fig. 5. As depicted in this figure, in both cases of sequence set design with good

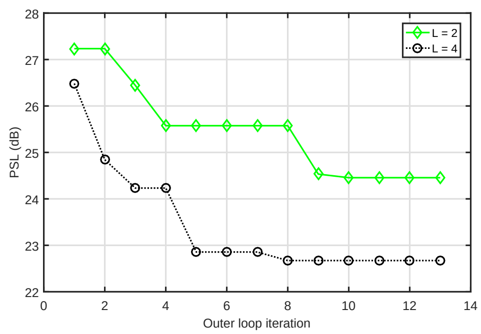
[8]The Matlab codes for Multi-CAN and Multi-PeCAN are downloaded from the book website http://www.sal.ufl.edu/book/.

[9]We use $\text{PSL}^{\mathcal{AP}}$ to show the PSL values corresponding to the minimization of aperiodic correlation functions whereas the PSL values related to the minimization of periodic correlation functions are depicted by $\text{PSL}^{\mathcal{P}}$.

[10]The PSL values corresponding to the starting set of random sequences are depicted in the figure.

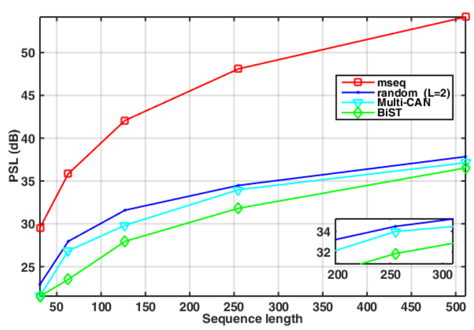


(a) Aperiodic correlation function.

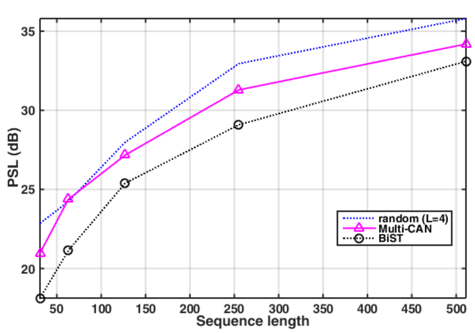


(b) Periodic correlation function.

Fig. 1: Convergence behavior of the BiST algorithm in PSL(dB) minimization.

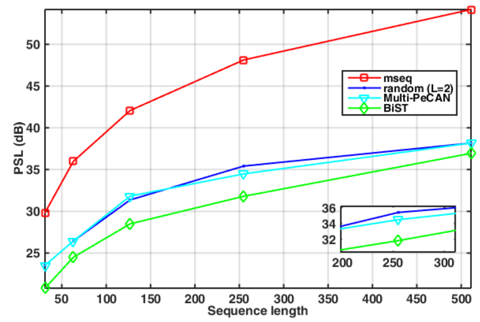


(a) Comparison between obtained $\text{PSL}^{\mathcal{AP}}$(dB) of aperiodic correlation functions for different sets of m-sequences, random, Multi-CAN and BiST ($L = 2$).

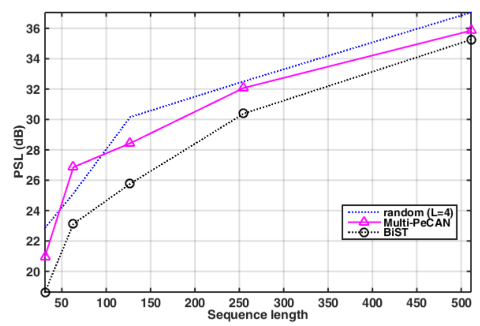


(b) Comparison between obtained $\text{PSL}^{\mathcal{AP}}$(dB) of aperiodic correlation functions for different sets of random sequences, Multi-CAN and BiST ($L = 4$).

Fig. 2: $\text{PSL}^{\mathcal{AP}}$(dB) versus sequence length.



(a) Comparison between obtained $\text{PSL}^{\mathcal{P}}$(dB) of periodic correlation functions for different sets of m-sequences, random, Multi-PeCAN and BiST ($L = 2$).



(b) Comparison between obtained $\text{PSL}^{\mathcal{P}}$(dB) of periodic correlation functions for different sets of random sequences, Multi-PeCAN and BiST ($L = 4$).

Fig. 3: $\text{PSL}^{\mathcal{P}}$(dB) versus sequence length.

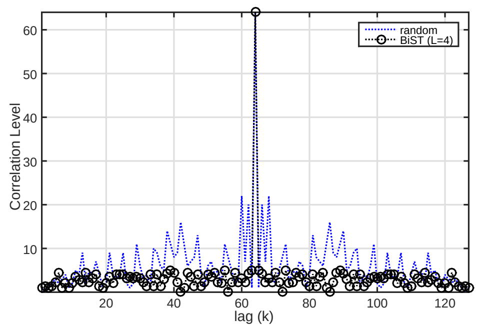
aperiodic (Fig. 5a) and periodic (Fig. 5b) correlation functions, BiST retains its monotonic decreasing behavior.

In Fig. 6, setting $N = \{31, 63, 127, 255, 511\}$ we asses the performance of the proposed method in ISL[11] minimization. The benchmark is Multi-CAN in case of sequence design with good aperiodic correlation function (Fig. 6) and is Multi-PeCAN when we design set of sequences with good periodic correlation function. Also, the ISL values corresponding to a set of m-sequences are plotted where the constellation size is $L = 2$ (Fig. 6a and Fig. 7a). As figures 6 and 7 show, BiST algorithm achieves lower ISL than its counterparts for all sequence lengths and at both constellation sizes $L = 2, 4$. As an example, the periodic auto- and cross-correlation functions of the obtained set of sequences for $N = 63$ and $L = 4$ are shown in Fig. 8a and Fig. 8b, respectively.

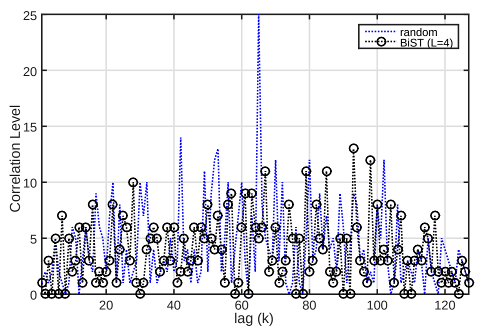
C. Pareto-Optimal Solution

In this subsection, the effect of the parameter w on the designed set of sequences is assessed. Table I reports PSL and ISL of the solutions obtained via BiST assuming $N = 63$, $w \in \{w_1, \ldots, w_5\}$ with $w_i = \frac{(i-1)}{4}$ and $i = 1, \ldots, 5$. The starting set of sequences is random when $w = w_1$, but the starting set of sequences at $w = w_i, i > 1$ is the optimized set

[11]We use $\text{ISL}^{\mathcal{AP}}$ to show the ISL values corresponding to the minimization of aperiodic correlation functions whereas the ISL values related to the minimization of periodic correlation functions are depicted by $\text{ISL}^{\mathcal{P}}$.

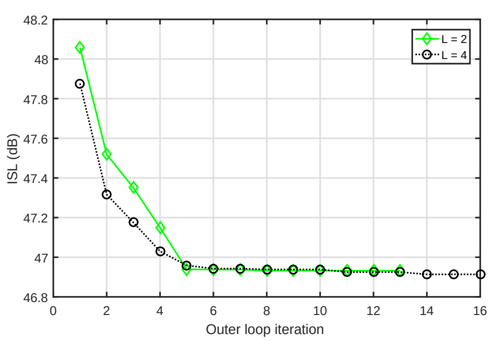


(a) Aperiodic auto-correlation of an obtained sequence via BiST setting $w = 1$ (PSL minimization).

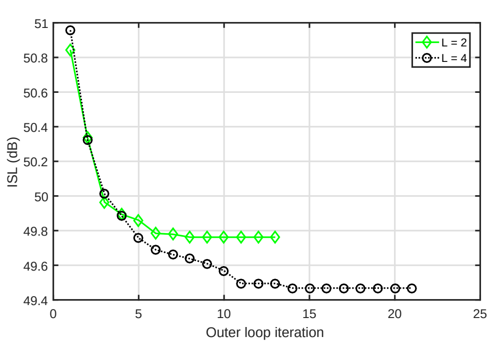


(b) Aperiodic cross-correlation of an obtained sequence via BiST setting $w = 1$ (PSL minimization).

Fig. 4: Correlation levels versus lag (k) of the sequences obtained via BiST ($L = 4$).

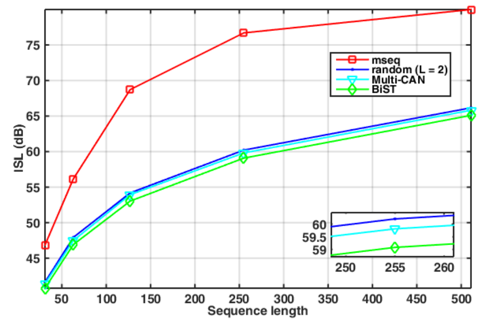


(a) Aperiodic correlation function.

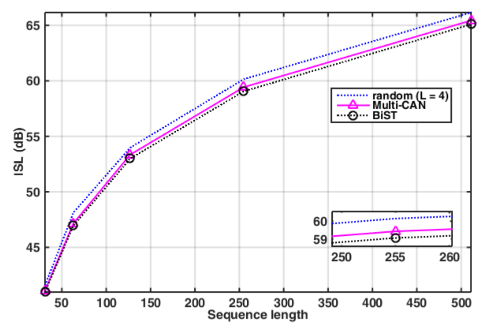


(b) Periodic correlation function.

Fig. 5: Convergence behavior of the BiST algorithm in ISL(dB) minimization.



(a) Comparison between obtained $\text{ISL}^{\mathcal{AP}}$(dB) of aperiodic correlation functions for different sets of m-sequences, random, Multi-CAN and BiST ($L = 2$).



(b) Comparison between obtained $\text{ISL}^{\mathcal{AP}}$(dB) of aperiodic correlation functions for different sets of random sequences, Multi-CAN and BiST ($L = 4$).

Fig. 6: $\text{ISL}^{\mathcal{AP}}$(dB) versus sequence length.

TABLE I: PSL and ISL of "Pareto-Optimal" Solutions (binary codes, $L = 2$)

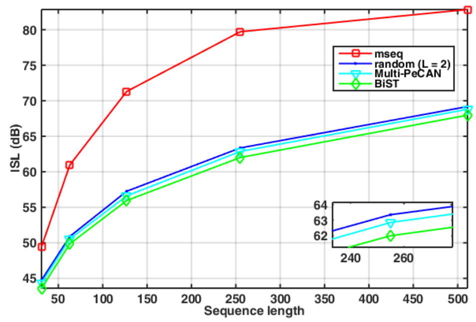
	$w = 0$	$w = 0.25$	$w = 0.5$	$w = 0.75$	$w = 1$
$\text{PSL}^{\mathcal{AP}}$ (dB)	26.84	25.98	25.02	24.05	23.52
$\text{ISL}^{\mathcal{AP}}$ (dB)	46.93	46.93	46.94	46.97	47.49
$\text{PSL}^{\mathcal{P}}$ (dB)	27.95	27.23	27.01	26.45	24.42
$\text{ISL}^{\mathcal{P}}$ (dB)	49.79	49.85	49.88	50.12	50.76

of sequences at $w = w_{i-1}$.

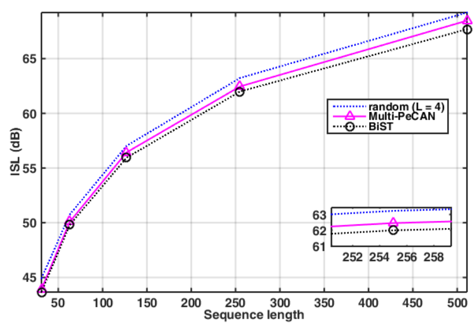
In Table I, we show the obtained values of PSL and ISL for both aperiodic and periodic correlation functions in the case of binary set of sequences ($L = 2$). The corresponding values for the case of $L = 4$ is written in Table II. As expected, w trades-off ISL and PSL values. Specifically, the higher the w the better the PSL while the worst the ISL. That is a classical feature of bi-objective Pareto curves.

TABLE II: PSL and ISL of "Pareto-Optimal" Solutions (L = 4)

	$w = 0$	$w = 0.25$	$w = 0.5$	$w = 0.75$	$w = 1$
$\text{PSL}^{\mathcal{AP}}$ (dB)	24.29	24.29	23.87	23.15	21.65
$\text{ISL}^{\mathcal{AP}}$ (dB)	46.92	46.95	46.95	46.97	47.09
$\text{PSL}^{\mathcal{P}}$ (dB)	26.82	25.98	25.42	25.22	22.59
$\text{ISL}^{\mathcal{P}}$ (dB)	49.45	49.77	49.81	50.11	50.35

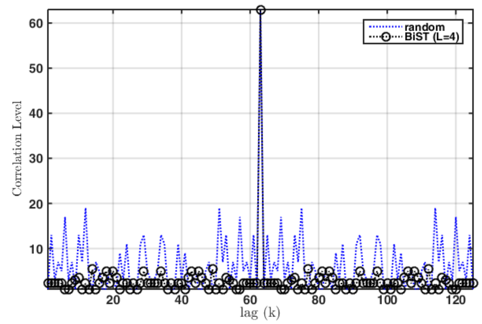


(a) Comparison between obtained $\mathrm{ISL}^{\mathcal{P}}$(dB) of periodic correlation functions for different sets of m-sequences, random, Multi-PeCAN and BiST ($L = 2$).

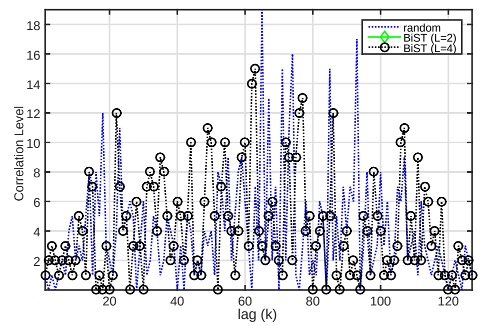


(b) Comparison between obtained $\mathrm{ISL}^{\mathcal{P}}$(dB) of periodic correlation functions for different sets of random sequences, Multi-PeCAN and BiST ($L = 4$).

Fig. 7: $\mathrm{ISL}^{\mathcal{P}}$(dB) versus sequence length.



(a) Periodic auto-correlation of an obtained sequence via BiST setting $w = 0$ (ISL minimization).



(b) Periodic cross-correlation of an obtained sequence via BiST setting $w = 0$ (ISL minimization).

Fig. 8: Correlation values versus lag (k) of the sequences obtained via BiST when $L = 4$.

VI. Conclusion

A computational approach to deal with the problem of set of phase sequences design with good aperiodic/periodic correlation functions for MIMO radar systems has been addressed in this paper. A general framework (called BiST) was devised to tackle the unimodular discrete phase sequence design problem. The results can be summarized as follows:

- The non-convex and, in general, NP-hard sequence design problem is handled via a novel two-step iterative procedure based on the CD method.
- Using the concept of coordinate descent algorithm, some basic formulations were provided that led to the proposed algorithm for minimization of a weighted sum of PSL and ISL of a set of sequences in a MIMO radar system.
- The provided numerical examples confirm that the BiST is an interesting approach for designing set of binary (discrete phase) sequences applicable in MIMO radar systems.

As future research tracks, it might be interesting to account for the behavior in the Doppler domain of the synthesized code, i.e., considering the design of a set of discrete phase sequences with a proper ambiguity function.

Appendix A
Proof of Lemma 1

The L-point DFT of ζ_{dkz} is,

$$\mathcal{F}_L(\zeta_{dkz}) = \begin{bmatrix} a_{dkz} + c_{dkz} + b_{dkz} \\ a_{dkz} + c_{dkz} e^{-\jmath \frac{2\pi}{L}} + b_{dkz} e^{-\jmath \frac{4\pi}{L}} \\ \vdots \\ a_{dkz} + c_{dkz} e^{-\jmath \frac{2\pi(L-1)}{L}} + b_{dkz} e^{-\jmath \frac{4\pi(L-1)}{L}} \end{bmatrix}$$

Next, observe that in term of auto-correlation we can write,

$$r_{tt,k}(\phi_t(d)) e^{-\jmath \phi_t(q)} = a_{dkt} + c_{dkt} e^{-\jmath \phi_t(q)} + b_{dkt} e^{-2\jmath \phi_t(q)}, \quad q = 1, \dots, L.$$

and for the cross-correlation function,

$$r_{tl,k}(\phi_t(d)) e^{-\jmath \phi_t(q)} = a_{dkl} + c_{dkl} e^{-\jmath \phi_t(q)}, \quad q = 1, \dots, L.$$

Since $|r_{tt,k}(\phi_t(d)) e^{-\jmath \phi_t(h)}| = |r_{tt,k}(\phi_t(d))|$ and $|r_{tl,k}(\phi_t(d)) e^{-\jmath \phi_t(h)}| = |r_{tl,k}(\phi_t(d))|$ we observe,

$$|\mathcal{F}_L(\zeta_{dkz})| = \Big[|r_{tz,k}(\phi_t(1))|, |r_{tz,k}(\phi_t(2))|, \dots, |r_{tz,k}(\phi_t(L))| \Big]^T, \quad (54)$$

which proofs $\boldsymbol{\nu}_{dkz} = |\mathrm{DFT}(\boldsymbol{\zeta}_{dkz})|^2$.

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