A Doppler Robust Design of Transmit Sequence and Receive Filter in the Presence of Signal-Dependent Interference

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Abstract

In this paper, we study the joint design of Doppler robust transmit sequence and receive filter to improve the performance of an active sensing system dealing with signal-dependent interference. The signal-to-noise-plus-interference (SINR) of the filter output is considered as the performance measure of the system. The design problem is cast as a max-min optimization problem to robustify the system SINR with respect to the unknown Doppler shifts of the targets. To tackle the design problem, which belongs to a class of NP-hard problems, we devise a novel method (which we call DESIDE) to obtain optimized pairs of transmit sequence and receive filter sharing the desired robustness property. The proposed method is based on a cyclic maximization of SINR expressions with relaxed rank-one constraints, and is followed by a novel synthesis stage. We devise synthesis algorithms to obtain high quality pairs of transmit sequence and receive filter that well approximate the behavior of the optimal SINR (of the relaxed problem) with respect to target Doppler shift. Several numerical examples are provided to analyze the performance obtained by DESIDE.

Keywords: Code design, Doppler shift, interference, receive filter, robust design, synthesis, transmit sequence

I. INTRODUCTION

The performance of an active sensing system can be significantly improved by judiciously designing its transmit sequence and receive filter. Such a design usually deals with several challenges including the fact that Doppler shifts of moving targets are often unknown at the transmit side, the existence of signal-dependent interference as well as signal-independent interference at the receive side, and practical constraints such as similarity to a given code.

Joint design of the transmit sequence and the receive filter has been considered in a large number of studies during the last decades. Most of the works have been concerned with either stationary targets or targets with known Doppler shifts (see e.g. [1]–[8]). In [9], considering a stationary target, a frequency domain approach has been employed to obtain an optimal receive filter and corresponding optimal energy spectral density of the transmit signal; then a synthesis procedure has been used to approximately provide the time domain signal. The works of [10] and [11] consider a related problem to that of [9] under a peak-to-average power ratio (PAR) constraint. The reference [12] deals with joint design of transmit sequence and receive filter under a similarity constraint in cases where the Doppler shift of the target is known. In [13], constant-modulus transmit sequences are considered in a framework similar to that of [12]. Several researches consider signal-independent clutter scenarios (see e.g. [14]–[18]). The unknown Doppler shift of the target has been taken into account in [16] and [18]. The reference [16] considers Doppler robust code design problem for signal-independent clutter cases under a similarity constraint. The ideas of [16] are generalized in [18] where the PAR constraint is also imposed.

In this paper, we devise a novel method for **D**oppler robust joint **de**sign of transmit sequence and receive filter (which we call DESIDE) in the presence of clutter. We focus on radar systems but the design methodology can be useful for other active sensing systems such as sonar, seismic exploration, etc. We consider the SINR at the output of the receive filter as the performance measure. Besides an energy constraint, a similarity constraint is imposed on the transmit sequence to control certain characteristics of the transmit waveform. The design problem is cast as a max-min optimization and shown to belong to a class of NP-hard problems. We devise a cyclic maximization to tackle a relaxed version of the design problem. Furthermore, we propose a synthesis stage to obtain optimized pairs of transmit sequences and receive filters which possess the desired Doppler robustness.

The rest of this paper is organized as follows. The data modeling and problem formulation are presented

in Section II. Section III contains the steps for the derivation of the cyclic approach to tackle the relaxed problem. The required synthesis stage is discussed in Section IV. Numerical results are provided in Section V. Finally, conclusions are drawn in Section VI.

Notation: We use bold lowercase letters for vectors and bold uppercase letters for matrices. $(\cdot)^T$, $(\cdot)^*$ and $(\cdot)^H$ denote the vector/matrix transpose, the complex conjugate, and the Hermitian transpose, respectively. $\mathbf{I}$ represents the identity matrix in $\mathbb{C}^{N \times N}$. $\mathbf{1}$ and $\mathbf{0}$ are the all-one and the all-zero vectors/matrices. $\mathbf{e}_k$ is the kth standard basis vector in $\mathbb{C}^N$. The l_2-norm of a vector $\mathbf{x}$ is denoted by $\|\mathbf{x}\|$. The symbol $\odot$ stands for the Hadamard (element-wise) product of matrices. $\mathrm{tr}(\cdot)$ is the trace of a square matrix argument. The notations $\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$ indicate the principal and the minor eigenvalues of a Hermitian matrix, respectively. $\mathbf{Diag}(\cdot)$ denotes the diagonal matrix formed by the entries of the vector argument, whereas $\mathbf{diag}(\cdot)$ denotes the vector formed by collecting the diagonal entries of the matrix argument. We write $\mathbf{A} \succeq \mathbf{B}$ iff $\mathbf{A} - \mathbf{B}$ is positive semi-definite, and $\mathbf{A} \succ \mathbf{B}$ iff $\mathbf{A} - \mathbf{B}$ is positive-definite. $\Re(\cdot)$ and $\arg(\cdot)$ denote the real-part and the phase angle (in radians) of the complex-valued argument. Finally, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ represent the set of natural, real and complex numbers, respectively.

II. Problem Formulation

We consider a radar system with (slow-time) transmit sequence $\mathbf{x} \in \mathbb{C}^N$ and receive filter $\mathbf{w} \in \mathbb{C}^N$. The discrete-time received signal backscattered from a moving target corresponding to the range-azimuth cell under the test can be modeled as (see, e.g. [12], [13], and [15]):

$$\mathbf{r} = \alpha_T \mathbf{x} \odot \mathbf{p}(\nu) + \mathbf{c} + \mathbf{n}, \tag{1}$$

where α_T is a complex parameter associated with backscattering effects of the target as well as propagation effects, $\mathbf{p}(\nu) = [1, e^{j\nu}, \ldots, e^{j(N-1)\nu}]^T$ with ν being the normalized target Doppler shift (expressed in radians), $\mathbf{c}$ is the N-dimensional column vector containing clutter (signal-dependent interference) samples, and $\mathbf{n}$ is the N-dimensional column vector of (signal-independent) interference samples. The vector $\mathbf{c}$ is the superposition of the returns from different uncorrelated scatterers located at various range-azimuth bins and can be expressed as [12]

$$\mathbf{c} = \sum_{k=0}^{N_c-1} \sum_{i=0}^{L-1} \alpha_{(k,i)} \mathbf{J}_k \left(\mathbf{s} \odot \mathbf{p}(\nu_{d_{(k,i)}}) \right)$$

where $N_c \leq N$ is the number of range rings[1] that interfere with the range-azimuth bin of interest $(0,0)$, L is the number of discrete azimuth sectors, $\alpha_{(k,i)}$ and $\nu_{d_{(k,i)}}$ denote the echo and the normalized Doppler shift, respectively, of the scatterer in the range-azimuth bin (k,i), and $\mathbf{J}_k$ denotes the aperiodic shifting matrix for $0 \leq k \leq N_c - 1$, viz.

$$\mathbf{J}_k(l,m) = \begin{cases} 1 & \text{if} \quad l-m = k \\ 0 & \text{if} \quad l-m \neq k \end{cases} \qquad (l,m) \in \{1,\dots,N\}^2$$

with $\mathbf{J}_{-k} = \mathbf{J}_k^T$.

The SINR at the output of the receive filter can be formulated as

$$\text{SINR}(\nu) = \frac{|\alpha_T|^2 \left|\mathbf{w}^H \left(\mathbf{x} \odot \mathbf{p}(\nu)\right)\right|^2}{\mathbf{w}^H \mathbf{\Sigma_c}\left(\mathbf{x}\right)\mathbf{w} + \mathbf{w}^H \mathbf{M}\mathbf{w}} \tag{2}$$

where $\mathbf{M} \triangleq \mathrm{E}\{\mathbf{n}\mathbf{n}^H\}$ and $\mathbf{\Sigma_c}\left(\mathbf{x}\right)$ is the covariance matrix of $\mathbf{c}$ given by [12]

$$\mathbf{\Sigma_c}\left(\mathbf{x}\right) = \sum_{k=0}^{N_c-1} \sum_{i=0}^{L-1} \sigma_{(k,i)}^2 \mathbf{J}_k \mathbf{\Gamma}(\mathbf{x},(k,i))\mathbf{J}_k^T \tag{3}$$

with $\sigma_{(k,i)}^2 = \mathrm{E}\left[|\alpha_{(k,i)}|^2\right]$ being the mean interfering power associated with the clutter patch located at the $(k,i)^{th}$ range-azimuth bin whose Doppler shift is supposed to be uniformly distributed in the interval $\Omega_c = \left(\bar{\nu}_{d_{(k,i)}} - \frac{\epsilon_{(k,i)}}{2}, \bar{\nu}_{d_{(k,i)}} + \frac{\epsilon_{(k,i)}}{2}\right)$ [15]. Herein $\mathbf{\Gamma}(\mathbf{x},(k,i)) = \mathrm{Diag}(\mathbf{x})\mathbf{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d_{(k,i)}}}\mathrm{Diag}(\mathbf{x})^H$ where $\mathbf{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d_{(k,i)}}}(l,m)$ is the covariance matrix of $\mathbf{p}(\nu_{d_{(k,i)}})$ [12], viz.

$$\mathbf{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d_{(k,i)}}}(l,m) = \begin{cases} 1 & \text{if} \quad l=m \\ e^{\left(j(l-m)\bar{\nu}_{d_{(k,i)}}\right)} \frac{\sin[0.5(l-m)\epsilon_{(k,i)}]}{[0.5(l-m)\epsilon_{(k,i)}]} & \text{if} \quad l \neq m \end{cases} \qquad (l,m) \in \{1,\dots,N\}^2. \tag{4}$$

Note that the expression for $\mathbf{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d_{(k,i)}}}(l,m)$ can be modified to consider cases with an arbitrary statistical distribution of the Doppler shifts of the clutter scatterers.

In this study we assume that the parameters of clutter and signal-independent interference are known at the transmit side by using cognitive (knowledge-aided) methods [12] [19]. We consider the SINR in (2) as the performance measure of the system [12] [15] and aim to find a robust design of the transmit sequence and the receive filter with respect to the unknown Doppler shift of the target[2]. In addition to

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[1]Note that the model considers the general case of range ambiguous clutter and reduces to unambiguous range scenario for $N_c = 1$. See [12] and [15] for justifications of the employed model and several examples of scenes that can be modeled in this way.

[2]The target Doppler shift can be estimated at the receiver, e.g. via a bank of filters matched to different Doppler frequencies [20]; however, the Doppler shifts of the targets are usually unknown at the transmit side and hence we consider a robust design with respect to the target Doppler shift. The design approach can also be useful for a robust confirmation process, so as to account for target Doppler estimation errors.

an energy constraint, a similarity constraint is imposed on the transmit sequence [21] [12] [22]:

$$\|\mathbf{x} - \mathbf{x}_0\|^2 \leq \delta, \tag{5}$$

where the parameter $\delta \geq 0$ rules the size of the similarity region and $\mathbf{x}_0$ is a given sequence. There are several reasons that justify the use of a similarity constraint in the design of a radar sequence. The unconstrained optimization of SINR can lead to signals with significant modulus variations, poor range resolution, high peak sidelobe levels, and more generally with an undesired ambiguity function behavior. These drawbacks can be partially circumvented imposing the similarity constraint (5) on the sought radar code [12] [21] [22]. Comprehensive simulations have been performed in [12] [16] [21] and [23] to illustrate how the properties of the ambiguity function (e.g. range resolution, sidelobe levels, etc.) and modulus variations associated with the optimized code can be controlled via the value of δ in the similarity constraint. By doing so, it is required that the solution be similar to a known sequence $\mathbf{x}_0$ which has some good properties such as constant modulus, reasonable range resolution, and peak sidelobe level.

The problem of Doppler robust joint design of transmit sequence $\mathbf{x}$ and receive filter $\mathbf{w}$ under the similarity constraint can be cast as the following max-min optimization problem

$$\mathcal{P} \begin{cases} \underset{\mathbf{x},\mathbf{w}}{\max} \; \underset{\nu}{\min} \; \dfrac{\left|\mathbf{w}^H \left(\mathbf{x} \odot \mathbf{p}(\nu)\right)\right|^2}{\mathbf{w}^H \mathbf{\Sigma_c}\left(\mathbf{x}\right)\mathbf{w} + \mathbf{w}^H \mathbf{M}\mathbf{w}} \\ \text{subject to} \quad \|\mathbf{x}\|^2 = e \\ \qquad\qquad\; \|\mathbf{x} - \mathbf{x}_0\|^2 \leq \delta \\ \qquad\qquad\; \nu \in \Omega \end{cases} \tag{6}$$

where $\Omega = [\nu_l, \nu_u] \subseteq [-\pi, \pi]$ denotes a given interval of the target Doppler shift ν and e denotes the maximum available transmit energy. Note that for a priori known target Doppler shift $\tilde{\nu}$ (i.e. $\Omega = [\tilde{\nu}, \tilde{\nu}]$), the problem $\mathcal{P}$ boils down to the considered problem in [12].

Remark 1: Note that a similar discrete-time data modeling and problem formulation applies to fast-time coding systems. In that case, the entries of $\mathbf{x}$ denote (complex) weights of the sub-pulses within a transmit pulse. Moreover, the normalized target Doppler shift ν is proportional to the system bandwidth (as opposed to the slow-time scheme for which ν is proportional to the pulse repetition frequency of the system); hence in such a case, the Doppler robust design would be concerned with high speed moving targets. As to the expressions, the formulation of the covariance matrix $\mathbf{\Sigma_c}\left(\mathbf{x}\right)$ in (3) should be modified. More precisely, for fast-time coding scenarios, the summation over k in (3) should be performed for $0 < |k| \leq N - 1$. We refer interested readers to the references [10] and [20] for more details on this aspect. ∎

To realize the hardness of the above problem, let z' and $\bar{\mathbf{x}}_\star$ denote a slack variable and an optimal solution $\mathbf{x}$ to the problem $\mathcal{P}$, respectively. The optimal $\mathbf{w}$ is obtained via solving the following optimization problem:

$$
\begin{cases}
\underset{\mathbf{w},z'}{\max} & \dfrac{z'}{\mathbf{w}^H \boldsymbol{\Sigma}_{\mathbf{c}}\left(\bar{\mathbf{x}}_\star\right)\mathbf{w} + \mathbf{w}^H \mathbf{M}\mathbf{w}} \\
\text{subject to} & \mathbf{w}^H \left(\bar{\mathbf{x}}_\star \bar{\mathbf{x}}_\star^H \odot \mathbf{p}(\nu)\mathbf{p}(\nu)^H\right)\mathbf{w} \geq z' \\
& \forall \nu \in \Omega.
\end{cases}
\tag{7}
$$

The above quadratic fractional program can be recast equivalently as (see Lemma 2 below and [24]):

$$
\mathcal{P}_{NP}
\begin{cases}
\underset{\mathbf{w},z'}{\max} & z' \\
\text{subject to} & \mathbf{w}^H \left(\boldsymbol{\Sigma}_{\mathbf{c}}\left(\bar{\mathbf{x}}_\star\right) + \mathbf{M}\right)\mathbf{w} \leq 1 \\
& \mathbf{w}^H \left(\bar{\mathbf{x}}_\star \bar{\mathbf{x}}_\star^H \odot \mathbf{p}(\nu)\mathbf{p}(\nu)^H\right)\mathbf{w} \geq z' \\
& \forall \nu \in \Omega.
\end{cases}
\tag{8}
$$

The optimization problem $\mathcal{P}_{NP}$ is a quadratically constrained quadratic program (QCQP) with infinitely many non-convex constraints. This class of QCQPs is known to be NP-hard in general [16] [25, Chapter 4] [26]. Note that solving the optimization problem $\mathcal{P}$ with respect to $(\mathbf{w}, \mathbf{x}, \nu)$ is at least as hard as solving the problem $\mathcal{P}_{NP}$.

The following lemma helps tackling the optimization problem $\mathcal{P}$ via providing two alternative expressions for the objective function in problem $\mathcal{P}$.

Lemma 1. *Let* $\mathbf{X} = \mathbf{x}\mathbf{x}^H$ *and* $\mathbf{W} = \mathbf{w}\mathbf{w}^H$. *The* $SINR(\nu)$ *can be alternatively expressed with respect to* $\mathbf{X}$ *and* $\mathbf{W}$ *as follows:*

$$
SINR(\nu) \;=\; \frac{|\alpha_T|^2 \mathbf{p}(\nu)^H \left(\mathbf{W} \odot \mathbf{X}^*\right)\mathbf{p}(\nu)}{\operatorname{tr}\left\{\left(\boldsymbol{\Sigma}_{\mathbf{c}}\left(\mathbf{X}\right) + \mathbf{M}\right)\mathbf{W}\right\}}
\tag{9}
$$

$$
\;=\; \frac{|\alpha_T|^2 \mathbf{p}(\nu)^H \left(\mathbf{W} \odot \mathbf{X}^*\right)\mathbf{p}(\nu)}{\operatorname{tr}\left\{\left(\boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{W}) + (\frac{\beta}{e})\mathbf{I}\right)\mathbf{X}\right\}}
\tag{10}
$$

where $\beta = \operatorname{tr}\{\mathbf{M}\mathbf{W}\}$, *and*

$$
\boldsymbol{\Sigma}_{\mathbf{c}}\left(\mathbf{X}\right) \;=\; \sum_{k=0}^{N_c-1} \sum_{i=0}^{L-1} \sigma_{(k,i)}^2 \mathbf{J}_k \left(\mathbf{X} \odot \boldsymbol{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d_{(k,i)}}}\right) \mathbf{J}_k^T,
\tag{11}
$$

$$
\boldsymbol{\Theta}_{\mathbf{c}}\left(\mathbf{W}\right) \;=\; \sum_{k=0}^{N_c-1} \sum_{i=0}^{L-1} \sigma_{(k,i)}^2 \left(\left(\mathbf{J}_k^T \mathbf{W} \mathbf{J}_k\right) \odot \left(\boldsymbol{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d_{(k,i)}}}\right)^*\right).
\tag{12}
$$

Proof: See Appendix A. ∎

To deal with the design problem $\mathcal{P}$, consider the following optimization problem:

$$
\mathcal{P}' \begin{cases}
\underset{\mathbf{X},\mathbf{W}}{\max}\ \underset{\nu}{\min} & \dfrac{\mathbf{p}(\nu)^H\left(\mathbf{W}\odot\mathbf{X}^*\right)\mathbf{p}(\nu)}{\operatorname{tr}\left\{\left(\boldsymbol{\Sigma_c}\left(\mathbf{X}\right)+\mathbf{M}\right)\mathbf{W}\right\}} \\[2mm]
\text{subject to} & \operatorname{tr}\{\mathbf{X}\}=e \\[1mm]
& \operatorname{tr}\{\mathbf{X}\mathbf{X}_0\}\geq\epsilon_\delta \\[1mm]
& \operatorname{rank}(\mathbf{X})=1 \\[1mm]
& \operatorname{rank}(\mathbf{W})=1 \\[1mm]
& \mathbf{X}\succeq\mathbf{0} \\[1mm]
& \mathbf{W}\succeq\mathbf{0} \\[1mm]
& \nu\in\Omega
\end{cases}
\tag{13}
$$

where $\mathbf{X}_0=\mathbf{x}_0\mathbf{x}_0^H$ and $\epsilon_\delta=((2e-\delta)/2)^2$. Let $(\mathbf{W},\mathbf{X})$ denote an optimal solution to the above problem. Using Lemma 1 and the results of [21], it can be easily verified that an optimal solution to $\mathcal{P}$ is given by $(\mathbf{w},\mathbf{x}\,e^{j\arg(\mathbf{x}^H\mathbf{x}_0)})$ with $\mathbf{W}=\mathbf{w}\mathbf{w}^H$ and $\mathbf{X}=\mathbf{x}\mathbf{x}^H$.

Now observe that both the objective function and the rank constraints in $\mathcal{P}'$ are non-convex. In addition, $\mathbf{p}(\nu)$ belongs to a non-convex set for $\nu\in\Omega$. In the sequel, we relax the rank-one constraints on $\mathbf{X}$ and $\mathbf{W}$ in $\mathcal{P}'$ to obtain the relaxed problem $\mathcal{P}_1$:

$$
\mathcal{P}_1 \begin{cases}
\underset{\mathbf{X},\mathbf{W}}{\max}\ \underset{\nu}{\min} & \dfrac{\mathbf{p}(\nu)^H\left(\mathbf{W}\odot\mathbf{X}^*\right)\mathbf{p}(\nu)}{\operatorname{tr}\left\{\left(\boldsymbol{\Sigma_c}\left(\mathbf{X}\right)+\mathbf{M}\right)\mathbf{W}\right\}} \\[2mm]
\text{subject to} & \operatorname{tr}\{\mathbf{X}\}=e \\[1mm]
& \operatorname{tr}\{\mathbf{X}\mathbf{X}_0\}\geq\epsilon_\delta \\[1mm]
& \mathbf{X}\succeq\mathbf{0} \\[1mm]
& \mathbf{W}\succeq\mathbf{0} \\[1mm]
& \nu\in\Omega.
\end{cases}
\tag{14}
$$

The expression $\frac{|\alpha_T|^2\mathbf{p}(\nu)^H(\mathbf{W}\odot\mathbf{X}^*)\mathbf{p}(\nu)}{\operatorname{tr}\{(\boldsymbol{\Sigma_c}(\mathbf{X})+\mathbf{M})\mathbf{W}\}}$ for rank-one $\mathbf{X}$ and $\mathbf{W}$ (i.e., $\mathbf{X}=\mathbf{x}\mathbf{x}^H$ and $\mathbf{W}=\mathbf{w}\mathbf{w}^H$) is equal to $SINR(\nu)$ (see Lemma 1). When the rank constraints are omitted (i.e., for arbitrary $\mathbf{X}\succeq\mathbf{0}$ and $\mathbf{W}\succeq\mathbf{0}$), the expression $\frac{|\alpha_T|^2\mathbf{p}(\nu)^H(\mathbf{W}\odot\mathbf{X}^*)\mathbf{p}(\nu)}{\operatorname{tr}\{(\boldsymbol{\Sigma_c}(\mathbf{X})+\mathbf{M})\mathbf{W}\}}$ may be used in lieu of $SINR(\nu)$ and it will be denoted by $\widetilde{SINR}_{relax}(\nu)$ in the following. $SINR(\nu)$ is the restriction of $\widetilde{SINR}_{relax}(\nu)$ over the space of the rank-one positive semi-definite matrices $\mathbf{X}$ and $\mathbf{W}$ (due to the relaxation of the rank-one constraints on $\mathbf{X}$ and $\mathbf{W}$). The optimization problem $\mathcal{P}_1$ is still non-convex and will be discussed in the next section.

III. THE PROPOSED METHOD TO TACKLE THE RELAXED PROBLEM $\mathcal{P}_1$

In this section, we devise a novel cyclic algorithm (which we call DESIDE-R as it deals with the relaxed version of the original problem) to tackle the non-convex optimization problem $\mathcal{P}_1$. In a cyclic

algorithm, the optimization variables are partitioned into two parts; then, by starting from an initial point, optimization is cyclically performed with respect to each part (while the another part is fixed) [27]. In the following, we consider the maximization problem $\mathcal{P}_1$ with respect to $(\mathbf{X}, \mathbf{W})$ where $\mathbf{X}$ and $\mathbf{W}$ are the two partitions. The obtained pair $(\mathbf{W}_\star, \mathbf{X}_\star)$ which maximizes $\widetilde{SINR}_{relax}(\nu)$ will be used later to synthesize the optimized transmit sequence/receive filter pair $(\mathbf{x}_\star, \mathbf{w}_\star)$. The synthesis stage is addressed in Section IV.

- *Optimal $\mathbf{X}$ for fixed $\mathbf{W}$*:

Let $\widetilde{t} \in \mathbb{R}$ denote a slack variable. For fixed $\mathbf{W}$, the optimization in (14) is equivalent to the following maximization problem:

$$\mathcal{P}_X \begin{cases} \underset{\mathbf{X}, \widetilde{t}}{\max} & \dfrac{\widetilde{t}}{\operatorname{tr}\left\{\left(\boldsymbol{\Theta}_\mathbf{c}(\mathbf{W}) + (\frac{\beta}{e})\mathbf{I}\right)\mathbf{X}\right\}} \\ \text{subject to} & \mathbf{p}(\nu)^H \left(\mathbf{W} \odot \mathbf{X}^*\right)\mathbf{p}(\nu) \geq \widetilde{t}, \;\; \forall \, \nu \in \Omega \\ & \operatorname{tr}\{\mathbf{X}\} = e \\ & \operatorname{tr}\{\mathbf{X}\mathbf{X}_0\} \geq \epsilon_\delta \\ & \mathbf{X} \succeq \mathbf{0}. \end{cases} \tag{15}$$

Note that the above problem is feasible and has a finite-valued objective function over the constraint set (see eq. (29)). Moreover, problem $\mathcal{P}_X$ is a linear-fractional maximization problem with infinitely many constraints (see the first constraint in (15)). Inspired by Charnes-Cooper transform for tackling linear fractional programs [28], we let $\mathbf{Y} = s\mathbf{X}$, $t = s\widetilde{t}$ for an auxiliary variable $s \geq 0$, and consider the following optimization problem:

$$\mathcal{P}'_X \begin{cases} \underset{\mathbf{Y}, t, s}{\max} & t \\ \text{subject to} & \operatorname{tr}\left\{\left(\boldsymbol{\Theta}_\mathbf{c}(\mathbf{W}) + (\frac{\beta}{e})\mathbf{I}\right)\mathbf{Y}\right\} = 1 \\ & \mathbf{p}(\nu)^H \left(\mathbf{W} \odot \mathbf{Y}^*\right)\mathbf{p}(\nu) \geq t, \;\; \forall \, \nu \in \Omega \\ & \operatorname{tr}\{\mathbf{Y}\} = e\,s \\ & \operatorname{tr}\{\mathbf{Y}\mathbf{X}_0\} \geq \epsilon_\delta\,s \\ & \mathbf{Y} \succeq \mathbf{0} \\ & s \geq 0. \end{cases} \tag{16}$$

Lemma 2. *The optimization problems $\mathcal{P}_X$ and $\mathcal{P}'_X$ are equivalent. More precisely, they share the same optimal values and their corresponding solutions can be uniquely obtained from each other.*

Proof: Let $(\mathbf{X}_\star, \widetilde{t}_\star)$ and $v(\mathcal{P}_X)$ denote an optimal solution and the optimal value of the problem $\mathcal{P}_X$, respectively. Note that $\operatorname{tr}\left\{\left(\boldsymbol{\Theta}_\mathbf{c}(\mathbf{W}) + (\frac{\beta}{e})\mathbf{I}\right)\mathbf{Y}\right\} > 0$ because $\beta > 0$. It is straightforward to verify

that

$$(\mathbf{Y}, t, s) = \left(\frac{\mathbf{X}_\star}{\operatorname{tr}\left\{ \left(\boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{W}) + (\frac{\beta}{e})\mathbf{I} \right) \mathbf{X}_\star \right\}}, \frac{\widetilde{t}_\star}{\operatorname{tr}\left\{ \left(\boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{W}) + (\frac{\beta}{e})\mathbf{I} \right) \mathbf{X}_\star \right\}}, \frac{1}{\operatorname{tr}\left\{ \left(\boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{W}) + (\frac{\beta}{e})\mathbf{I} \right) \mathbf{X}_\star \right\}} \right) \tag{17}$$

is feasible for the problem $\mathcal{P}'_X$. Also observe that the value of the objective function of $\mathcal{P}'_X$ for $(\mathbf{Y}, t, s)$ in (17) is given by

$$\frac{t_\star}{\operatorname{tr}\left\{ \left(\boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{W}) + (\frac{\beta}{e})\mathbf{I} \right) \mathbf{X}_\star \right\}} \tag{18}$$

and note that (18) is equal to $v(\mathcal{P}_X)$. Therefore, for the optimal value of the problem $\mathcal{P}'_X$, i.e. $v(\mathcal{P}'_X)$, we have

$$v(\mathcal{P}'_X) \geq v(\mathcal{P}_X). \tag{19}$$

Next let $(\mathbf{Y}_\star, t_\star, s_\star)$ denote an optimal solution to the problem $\mathcal{P}'_X$. Note that $s_\star \neq 0$ because $s_\star = 0$ leads to $\mathbf{Y}_\star = \mathbf{0}$ (a contradiction, see the first constraint in $\mathcal{P}'_X$). One can check that $(\mathbf{Y}_\star/s_\star, t_\star/s_\star)$ is feasible for the problem $\mathcal{P}_X$ with corresponding objective value equal to $t_\star$. Owing to the fact that $v(\mathcal{P}'_X) = t_\star$, the following inequality holds between $v(\mathcal{P}'_X)$ and $v(\mathcal{P}_X)$:

$$v(\mathcal{P}'_X) \leq v(\mathcal{P}_X). \tag{20}$$

Finally, eqs. (19) and (20) yield $v(\mathcal{P}'_X) = v(\mathcal{P}_X)$ and the proof is concluded. ∎

Now observe that $\mathcal{P}'_X$ is a convex problem with infinitely many constraints. To deal with the constraint set, we note that the constraint $\mathbf{p}(\nu)^H \left(\mathbf{W} \odot \mathbf{Y}^* \right) \mathbf{p}(\nu) \geq t, \ \forall \nu \in \Omega$ implies the non-negativity of a trigonometric polynomial of ν over the interval Ω. More specifically, let

$$z_k \triangleq \sum_{i=1}^{N-k} Z_{i+k,i}, \quad 0 \leq k \leq N-1, \tag{21}$$

and $\mathbf{z} = [z_0, z_1, \ldots, z_{N-1}]^T$ with $\mathbf{Z} = \mathbf{W} \odot \mathbf{Y}^*$. It is straightforward to verify that for any $\nu \in \Omega$, the aforementioned constraint is equivalent to

$$h(\nu) \triangleq z_0 - t + 2\Re \left(\sum_{k=1}^{N-1} z_k e^{-jk\nu} \right) \geq 0. \tag{22}$$

Interestingly, a semidefinite representation of the constraint (22) can be obtained via Theorem 3.4 in [29] which we quote below.

Theorem 1. *The trigonometric polynomial $\tilde{h}(\nu) = \widetilde{z}_0 + 2\Re \left(\sum_{k=1}^{N-1} \widetilde{z}_k e^{-jk\nu} \right)$ is non-negative for any $\nu \in [\nu_0 - \nu_1, \nu_0 + \nu_1]$ (with $0 < \nu_1 < \pi$) iff there exist an $N \times N$ Hermitian matrix $\mathbf{Z}_1 \succeq \mathbf{0}$ and an*

$(N-1) \times (N-1)$ *Hermitian matrix* $\mathbf{Z}_2 \succeq \mathbf{0}$ *such that*

$$\widetilde{\mathbf{z}} = \mathbf{F}_1^H \left(\mathrm{diag}(\mathbf{F}_1 \mathbf{Z}_1 \mathbf{F}_1^H) + \mathbf{q} \odot \mathrm{diag}(\mathbf{F}_2 \mathbf{Z}_2 \mathbf{F}_2^H) \right) \tag{23}$$

where $\widetilde{\mathbf{z}} = [\widetilde{z}_0, \widetilde{z}_1, \ldots, \widetilde{z}_{N-1}]^T$, $\mathbf{q} = [q_0, q_1, \ldots, q_{n-1}]^T$ *with* $q_k = \cos(2\pi k/n - \nu_0) - \cos(\nu_1)$, $\mathbf{F}_1 = [\mathbf{f}_0, \ldots, \mathbf{f}_{N-1}]$ *and* $\mathbf{F}_2 = [\mathbf{f}_0, \ldots, \mathbf{f}_{N-2}]$ *in which* $\mathbf{f}_k = [1, e^{-jk\theta}, \ldots, e^{-j(n-1)k\theta}]^T$ *with* $\theta = 2\pi/n$, *and* $n \geq 2N - 1$.

Note that an SDP representation of (22) is immediate by employing the above results with $\widetilde{\mathbf{z}} = \mathbf{z}$, $n = 2N - 1$, $\nu_0 = (\nu_l + \nu_u)/2$, and $\nu_1 = \nu_0 - \nu_l$. Consequently, $\mathcal{P}'_X$ is equivalent to the following SDP:

$$\mathcal{SDP}_X \begin{cases} \displaystyle\max_{\mathbf{Y}, \mathbf{Z}_1, \mathbf{Z}_2, t, s} & t \\ \text{subject to} & \mathrm{tr}\left\{ \left(\boldsymbol{\Theta}_\mathbf{c}(\mathbf{W}) + (\frac{\beta}{e})\mathbf{I} \right) \mathbf{Y} \right\} = 1 \\ & \mathbf{z} = t\mathbf{e}_1 + \mathbf{F}_1^H \left(\mathrm{diag}(\mathbf{F}_1 \mathbf{Z}_1 \mathbf{F}_1^H) + \mathbf{q} \odot \mathrm{diag}(\mathbf{F}_2 \mathbf{Z}_2 \mathbf{F}_2^H) \right) \\ & \mathrm{tr}\{\mathbf{Y}\} = e\,s \\ & \mathrm{tr}\{\mathbf{Y}\mathbf{X}_0\} \geq \epsilon_\delta\,s \\ & \mathbf{Y} \succeq \mathbf{0} \\ & \mathbf{Z}_1 \succeq \mathbf{0} \\ & \mathbf{Z}_2 \succeq \mathbf{0} \\ & s \geq 0. \end{cases} \tag{24}$$

Remark 2: The derivation of $\mathcal{SDP}_X$ can be extended to deal with cases where Ω is a union of several (non-overlapping) sub-intervals of $[-\pi, \pi]$. More precisely, for each of such sub-intervals, the SDP representation associated with the corresponding constraint (obtained via Theorem 1) can be added to the constraint set of $\mathcal{SDP}_X$. ∎

Let $(\mathbf{Y}, \mathbf{Z}_1, \mathbf{Z}_2, t, s)$ denote an optimal solution to $\mathcal{SDP}_X$. The corresponding optimal $\mathbf{X}$ (i.e., an optimal solution to $\mathcal{P}_X$) for fixed $\mathbf{W}$ is given by $\mathbf{Y}/s$ (see Lemma 2).

- *Optimal* $\mathbf{W}$ *for fixed* $\mathbf{X}$:

Using Lemma 1, $\mathcal{P}_1$ can be recast into the following equivalent form for fixed $\mathbf{X}$:

$$\mathcal{P}_W \begin{cases} \displaystyle\max_{\mathbf{W}, \breve{t}} & \dfrac{\breve{t}}{\mathrm{tr}\left\{ (\boldsymbol{\Sigma}_\mathbf{c}(\mathbf{X}) + \mathbf{M})\,\mathbf{W} \right\}} \\ \text{subject to} & \mathbf{p}(\nu)^H \left(\mathbf{W} \odot \mathbf{X}^* \right) \mathbf{p}(\nu) \geq \breve{t}, \ \ \forall\,\nu \in \Omega \\ & \mathbf{W} \succeq \mathbf{0} \end{cases} \tag{25}$$

where $\breve{t}$ denotes a slack variable. The above problem can be tackled in a way similar to the case of obtaining $\mathbf{X}$ for fixed $\mathbf{W}$. In particular, using Lemma 2 as well as Theorem 1, we obtain the following

SDP:

$$\mathcal{SDP}_W \begin{cases} \max_{\mathbf{W},\mathbf{Z}'_1,\mathbf{Z}'_2,\breve{t}} & \breve{t} \\ \text{subject to} & \text{tr}\left\{(\boldsymbol{\Sigma}_{\mathbf{c}}(\mathbf{X}) + \mathbf{M})\,\mathbf{W}\right\} = 1 \\ & \mathbf{z}' = \breve{t}\mathbf{e}_1 + \mathbf{F}_1^H\left(\text{diag}(\mathbf{F}_1\mathbf{Z}'_1\mathbf{F}_1^H) + \mathbf{q}\odot\text{diag}(\mathbf{F}_2\mathbf{Z}'_2\mathbf{F}_2^H)\right) \\ & \mathbf{W} \succeq \mathbf{0} \\ & \mathbf{Z}'_1 \succeq \mathbf{0} \\ & \mathbf{Z}'_2 \succeq \mathbf{0} \end{cases} \tag{26}$$

where $\mathbf{z}'$ is given by

$$z'_k = \sum_{i=1}^{N-k} Z'_{i+k,i}, \quad 0 \le k \le N-1, \tag{27}$$

with $\mathbf{Z}' = \mathbf{W} \odot \mathbf{X}^*$.

Remark 3: It might be interesting in practice to control the shape of the cross-ambiguity function of the transmit sequence $\mathbf{x}$ and the receive filter $\mathbf{w}$. An approach would then be to require that the variables $\mathbf{w}$ and $\mathbf{x}$ are sufficiently similar to given $\mathbf{w}_0$ and $\mathbf{x}_0$, respectively, which possess desirable cross-ambiguity properties. The Doppler robust design for controlling the shape of the cross-ambiguity function could therefore be cast as the following optimization problem:

$$\mathcal{P}_{cross} \begin{cases} \max_{\mathbf{x},\mathbf{w}} \min_{\nu} & \dfrac{\left|\mathbf{w}^H\left(\mathbf{x}\odot\mathbf{p}(\nu)\right)\right|^2}{\mathbf{w}^H\boldsymbol{\Sigma}_{\mathbf{c}}(\mathbf{x})\,\mathbf{w} + \mathbf{w}^H\mathbf{M}\mathbf{w}} \\ \text{subject to} & \|\mathbf{x} - \mathbf{x}_0\|^2 \le \delta \\ & \|\mathbf{w} - \mathbf{w}_0\|^2 \le \delta_w \\ & \|\mathbf{x}\|^2 = e \\ & \nu \in \Omega \end{cases} \tag{28}$$

where δ_w rules the size of the similarity region for the receive filter. The problem $\mathcal{P}_{cross}$ can be tackled in a way similar to (6). ∎

The steps of DESIDE-R are summarized in Table I. Each iteration of the proposed method is handled via solving two SDPs, i.e., $\mathcal{SDP}_W$ and $\mathcal{SDP}_X$. The complexity of solving the SDPs with accuracy of ϵ_a is given by $\mathcal{O}(N^{3.5}\log(\epsilon_a^{-1}))$ [30]. A synthesis stage is proposed in the next section to compute high quality transmit sequence/receive filter pairs $(\mathbf{w}_\star, \mathbf{x}_\star)$ from the solutions $(\mathbf{W}_\star, \mathbf{X}_\star)$ obtained herein.

- *Convergence and the $\widetilde{SINR}_{relax}$ metric*:

By cyclically solving $\mathcal{SDP}_X$ and $\mathcal{SDP}_W$ in DESIDE-R, it can be easily verified that the resulting $\{\min_{\nu\in\Omega}\widetilde{SINR}_{relax}^{(\kappa)}(\nu)\}_{\kappa\in\mathbb{N}}$ is a monotonically increasing sequence [27]. Furthermore, $\min_{\nu\in\Omega}\widetilde{SINR}_{relax}(\nu)$

TABLE I

DESIDE-R METHOD FOR SOLVING THE RELAXED PROBLEM $\mathcal{P}_1$

Step 0: Initialize $\mathbf{X}$ with $\mathbf{x}\mathbf{x}^H$ where $\mathbf{x}$ is a random vector in $\mathbb{C}^N$.

Step 1: Solve the problem $\mathcal{SDP}_W$ in (26) to obtain $\mathbf{W}$.

Step 2: Solve the problem $\mathcal{SDP}_X$ in (24) to obtain $\mathbf{X}$.

Step 3: Repeat steps 1 and 2 until a pre-defined stop criterion is satisfied,

e.g. $|\min_{\nu \in \Omega} \widetilde{SINR}_{relax}(\nu)^{(\kappa+1)} - \min_{\nu \in \Omega} \widetilde{SINR}_{relax}(\nu)^{(\kappa)}| \leq \mu$ for

a given $\mu > 0$.

---

is bounded from above; indeed we have that

$$
\begin{aligned}
\min_{\nu \in \Omega} \widetilde{SINR}_{relax}(\nu) \quad &\leq \quad \frac{\mathbf{p}(\nu)^H \left(\mathbf{W} \odot \mathbf{X}^*\right)\mathbf{p}(\nu)}{\operatorname{tr}\{\mathbf{W}\left(\mathbf{\Sigma_c}(\mathbf{x}) + \mathbf{M}\right)\}} \\
&\leq \quad \frac{\|\mathbf{p}(\nu)\|^2 \lambda_{max}(\mathbf{W} \odot \mathbf{X}^*)}{\operatorname{tr}\{\mathbf{MW}\}} \\
&\leq \quad \frac{N \operatorname{tr}\{\mathbf{W}\}\operatorname{tr}\{\mathbf{X}\}}{\operatorname{tr}\{\mathbf{MW}\}} \\
&\leq \quad \frac{N e}{\lambda_{min}(\mathbf{M})}.
\end{aligned}
\tag{29}
$$

The third inequality above holds true because $\operatorname{tr}\{\mathbf{W} \odot \mathbf{X}^*\} \leq \operatorname{tr}\{\mathbf{W}\}\operatorname{tr}\{\mathbf{X}\}$; and for the last inequality we have used the fact that $\operatorname{tr}\{\mathbf{MW}\} \geq \lambda_{min}(\mathbf{M})\operatorname{tr}\{\mathbf{W}\}$ [31]. Eq. (29) along with the increasing property of $\{\min_{\nu \in \Omega} \widetilde{SINR}_{relax}^{(\kappa)}(\nu)\}_{\kappa \in \mathbb{N}}$ ensure the convergence of the sequence of the objective function values.

IV. THE SYNTHESIS STAGE

As discussed earlier, a judicious synthesis of the optimized transmit sequence $\mathbf{x}_\star$ and receive filter $\mathbf{w}_\star$ from the obtained $(\mathbf{W}_\star, \mathbf{X}_\star)$ is required to maintain the Doppler robustness. If $\mathbf{W}_\star$ is rank-one, $\mathbf{w}_\star$ is available via considering $\mathbf{W}_\star = \mathbf{w}_\star \mathbf{w}_\star^H$; whereas if $\mathbf{X}_\star = \mathbf{x}\mathbf{x}^H$, for $\mathbf{x}_\star$ we have $\mathbf{x}_\star = \mathbf{x}e^{j \arg(\mathbf{x}^H \mathbf{x}_0)}$. The rank behavior of SDP solutions, tightness of the semidefinite relaxation, and synthesis methods have been investigated in the literature (see e.g. [33]–[35], and references therein). For example, it is known that for a solvable[3] SDP with M constraints, there exists an optimal solution of rank at most equal to $\sqrt{M}$ [33]. However, the result does not ensure the existence of rank-one solutions for the case considered in this paper due to the fact that $\mathcal{SDP}_X$ and $\mathcal{SDP}_W$ have $N + 3$ and $N + 1$ constraints, respectively. Herein we remark on the fact that the ranks of $\mathbf{W}_\star$ and $\mathbf{X}_\star$ depend on the employed starting point in

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[3]Meaning that the SDP is feasible, bounded, and its optimal value is attained (see [30] for more details).

addition to the parameters of the design problem. In cases where the rank of either $\mathbf{W}_\star$ or $\mathbf{X}_\star$ is larger than one, the synthesis of $\mathbf{w}_\star$ or $\mathbf{x}_\star$ is more complicated. To tackle this problem, this section initially considers the rank-one decomposition method [36]. Then we devise novel synthesis algorithms to design pairs of $(\mathbf{w}_\star, \mathbf{x}_\star)$ possessing the desired robustness.

A. The rank-one decomposition method

The main result of the rank-one decomposition method can be summarized as follows [36]:

Theorem 2. *Let $\mathbf{X}$ be a non-zero $N \times N$ complex Hermitian positive semidefinite matrix (with $N \geq 3$) and $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ be Hermitian matrices. Suppose that $(\operatorname{tr}\{\mathbf{YA}_1\}, \operatorname{tr}\{\mathbf{YA}_2\}, \operatorname{tr}\{\mathbf{YA}_3\}, \operatorname{tr}\{\mathbf{YA}_4\}) \neq (0,0,0,0)$ for any non-zero complex Hermitian positive semidefinite matrix $\mathbf{Y}$ of size $N \times N$. Then,*

- *if $rank(\mathbf{X}) \geq 3$, one can find, in polynomial time, a rank-one matrix $\mathbf{xx}^H$ such that $\mathbf{x}$ (synthetically denoted as $\mathbf{x} = \mathcal{D}_1(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4)$) is in the range($\mathbf{X}$), and*

$$\mathbf{x}^H \mathbf{A}_i \mathbf{x} = \operatorname{tr}\{\mathbf{XA}_i\}, \quad i = 1,2,3,4;$$

- *if $rank(\mathbf{X}) = 2$, for any $\mathbf{z}$ not in the range space of $\mathbf{X}$, one can find a rank-one matrix $\mathbf{xx}^H$ such that $\mathbf{x}$ (synthetically denoted as $\mathbf{x} = \mathcal{D}_2(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4)$) is in the linear subspace spanned by $\{\mathbf{z}\} \cup range(\mathbf{X})$, and*

$$\mathbf{x}^H \mathbf{A}_i \mathbf{x} = \operatorname{tr}\{\mathbf{XA}_i\}, \quad i = 1,2,3,4.$$

Proof: see [36, Theorem 2.3]. ∎

Let $(\mathbf{W}_\star, \mathbf{X}_\star)$ denote an optimal solution to $\mathcal{P}_1$, and let

$$\nu_\star = \operatorname{argmin}_{\nu \in \Omega} \mathbf{p}(\nu)^H (\mathbf{W}_\star \odot \mathbf{X}_\star^*) \mathbf{p}(\nu). \tag{30}$$

Considering Theorem 2 and the problem $\mathcal{P}_W$, a suitable rank-one matrix $\mathbf{w}_\star \mathbf{w}_\star^H$ can be found such that

$$\operatorname{tr}\{\underbrace{(\mathbf{\Sigma}_c(\mathbf{X}_\star) + \mathbf{M})}_{\mathbf{Q}_1} \mathbf{W}_\star\} = \mathbf{w}_\star^H \mathbf{Q}_1 \mathbf{w}_\star \tag{31}$$

and that

$$\begin{cases} \operatorname{tr}\{\underbrace{(\mathbf{X}_\star \odot (\mathbf{p}(\nu_\star)\mathbf{p}(\nu_\star)^H))}_{\mathbf{Q}_2} \mathbf{W}_\star\} = \mathbf{w}_\star^H \mathbf{Q}_2 \mathbf{w}_\star \\ \operatorname{tr}\{\underbrace{(\mathbf{X}_\star \odot (\mathbf{p}(\nu')\mathbf{p}(\nu')^H))}_{\mathbf{Q}_3} \mathbf{W}_\star\} = \mathbf{w}_\star^H \mathbf{Q}_3 \mathbf{w}_\star \\ \operatorname{tr}\{\underbrace{(\mathbf{X}_\star \odot (\mathbf{p}(\nu'')\mathbf{p}(\nu'')^H))}_{\mathbf{Q}_4} \mathbf{W}_\star\} = \mathbf{w}_\star^H \mathbf{Q}_4 \mathbf{w}_\star \end{cases} \tag{32}$$

where ν' and ν'' are two arbitrary Doppler shifts in Ω. The equations in (32) have been written using the identity $\mathbf{p}(\nu)^H (\mathbf{W}_\star \odot \mathbf{X}_\star^*) \mathbf{p}(\nu) = \mathrm{tr}\{(\mathbf{X}_\star \odot (\mathbf{p}(\nu)\mathbf{p}(\nu)^H)) \mathbf{W}_\star\}$. Note that Theorem 2 lays the ground for considering two more Doppler frequencies (i.e. ν' and ν'') other than $\nu_\star$. This leads to a receive filter $\mathbf{w}_\star$ with a behavior more similar to that of $\mathbf{W}_\star$ with respect to target Doppler shift ν. Consequently, $\mathbf{w}_\star$ is obtained via $\mathbf{w}_\star = \mathcal{D}_1(\mathbf{W}_\star, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3, \mathbf{Q}_4)$ or $\mathbf{w}_\star = \mathcal{D}_2(\mathbf{W}_\star, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3, \mathbf{Q}_4)$ for cases where $\mathrm{rank}(\mathbf{W}_\star) \geq 3$ or $\mathrm{rank}(\mathbf{W}_\star) = 2$, respectively. Note that the condition $(\mathrm{tr}(\mathbf{Y}\mathbf{Q}_1), \mathrm{tr}(\mathbf{Y}\mathbf{Q}_2), \mathrm{tr}(\mathbf{Y}\mathbf{Q}_3), \mathrm{tr}(\mathbf{Y}\mathbf{Q}_4)) \neq (0,0,0,0)$ on the matrices $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$, and $\mathbf{Q}_4$ in Theorem 2 is satisfied; more precisely, there exists $(a_1, a_2, a_3, a_4) \in \mathbb{R}_+^4$ such that $a_1\mathbf{Q}_1 + a_2\mathbf{Q}_2 + a_3\mathbf{Q}_3 + a_4\mathbf{Q}_4 \succ \mathbf{0}$ (see [37]).

The $\mathbf{x}_\star$ can be obtained in a similar way; particularly, $\mathbf{x}_\star = \mathcal{D}_\zeta(\mathbf{X}_\star, \mathbf{R}_1, \mathbf{R}_2, \mathbf{X}_0, \mathbf{I})$ $(\zeta = 1$ or 2 depending on the rank of $\mathbf{X}_\star$) where

$$\mathbf{R}_1 = \mathbf{\Theta}_c(\mathbf{W}_\star) + (\beta/e)\mathbf{I} \tag{33}$$

$$\mathbf{R}_2 = \mathbf{W}_\star \odot \left((\mathbf{p}(\nu_\star)\mathbf{p}(\nu_\star)^H)^*\right). \tag{34}$$

B. New algorithms for synthesis stage

As explained earlier, the rank-one decomposition method can deal with at most four trace equalities for the synthesis of the receive filter and the transmit sequence. This ability allows for considering the values of the $\widetilde{SINR}_{relax}(\nu)$ at three Doppler shifts ν for the receive filter synthesis (and one Doppler shift for synthesis of the transmit sequence). However, the pair $(\mathbf{w}_\star, \mathbf{x}_\star)$ obtained by applying Theorem 2 can lead to $SINR(\nu)$ whose behavior with respect to target Doppler shift is not "sufficiently" close to the behavior of the $\widetilde{SINR}_{relax}(\nu)$. It means that the $SINR(\nu)$ can have significantly lower minimum value with respect to ν. In this subsection, we devise novel algorithms to synthesize high quality $\mathbf{w}_\star$ and $\mathbf{x}_\star$ from the solutions to the problem $\mathcal{P}_1$, i.e. $\mathbf{W}_\star$ and $\mathbf{X}_\star$. The idea is to consider the values of $\mathbf{p}(\nu)^H (\mathbf{W}_\star \odot \mathbf{X}_\star^*) \mathbf{p}(\nu)$ as the optimal energy spectral density (ESD) associated with the transmit sequence and the receive filter. The sought receive filter $\mathbf{w}_\star$ and transmit sequence $\mathbf{x}_\star$ are obtained to approximate well the behavior of the optimal ESD with respect to ν. Due to the fact that $\widetilde{SINR}_{relax}(\nu)$ (of $(\mathbf{W}_\star, \mathbf{X}_\star)$) is a scaled version of the optimal ESD (see (9)), we deal with the denominator of the $\widetilde{SINR}_{relax}(\nu)$ by imposing constraints in the synthesis problems.

Let $\{\nu_1, \nu_2, \cdots, \nu_K\}$ denote a discrete set of target Doppler shifts "uniformly distributed" over Ω, and consider the following quantities:

$$g_k \triangleq \mathbf{p}_k^H (\mathbf{W}_\star \odot \mathbf{X}_\star^*) \mathbf{p}_k \in \mathbb{R}_+, \ 1 \leq k \leq K \tag{35}$$

where $\mathbf{p}_k = \mathbf{p}(\nu_k)$. We define the vector $\mathbf{g} = [g_1, g_2, \cdots, g_K]^T$ as the optimal ESD.

- *Receive filter synthesis:* Herein the aim is to synthesize the optimized receive filter for given $(\mathbf{W}_\star, \mathbf{X}_\star)$. Observe that

$$\mathbf{p}_k^H \left(\mathbf{w}\mathbf{w}^H \odot \mathbf{X}_\star^* \right) \mathbf{p}_k = \mathbf{w}^H \underbrace{\left(\mathbf{p}_k \mathbf{p}_k^H \odot \mathbf{X}_\star \right)}_{\mathbf{T}_k} \mathbf{w}. \tag{36}$$

Note also that $\mathbf{T}_k \succeq \mathbf{0}$ for all k and so that there must exist $\mathbf{V}_k$ (of rank d_k) such that $\mathbf{T}_k = \mathbf{V}_k \mathbf{V}_k^H$. Moreover, considering $\mathbf{w}^H \mathbf{V}_k \mathbf{V}_k^H \mathbf{w} \approx g_k$, we can write

$$\mathbf{V}_k^H \mathbf{w} \approx \sqrt{g_k} \mathbf{u}_k, \ 1 \le k \le K, \tag{37}$$

where all $\mathbf{u}_k \in \mathbf{C}^{d_k}$ are unit-norm. Therefore, the receive filter $\mathbf{w}_\star$ can be found as the minimizer of the following metric:

$$\|\mathbf{A}\mathbf{w} - \mathbf{u} \odot \mathbf{b}\|^2 \tag{38}$$

where $\mathbf{A}^H = [\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_K]$, $\mathbf{u} = [\mathbf{u}_1^T, \mathbf{u}_2^T, \cdots, \mathbf{u}_K^T]^T$, and $\mathbf{b} = [\sqrt{g_1}\mathbf{1}^T, \sqrt{g_2}\mathbf{1}^T, \cdots, \sqrt{g_K}\mathbf{1}^T]^T$. Note that the optimal $\widetilde{SINR}_{relax}(\nu)$ (corresponding to $(\mathbf{W}_\star, \mathbf{X}_\star)$) is a scaled version of the optimal ESD for given $(\mathbf{W}_\star, \mathbf{X}_\star)$ (see (9)). As a result, to obtain the receive filter that yields $SINR(\nu)$ values close to $\widetilde{SINR}_{relax}(\nu)$, we should also take into account the denominator of the $\widetilde{SINR}_{relax}(\nu)$, viz.

$$\gamma = \mathrm{tr}\{(\boldsymbol{\Sigma}_{\mathbf{c}}(\mathbf{X}_\star) + \mathbf{M})\mathbf{W}_\star\}. \tag{39}$$

Consequently, we consider the following optimization problem to obtain $\mathbf{w}_\star$:

$$\mathcal{P}_w^{synt} \begin{cases} \min\limits_{\mathbf{w},\mathbf{u}} & \|\mathbf{A}\mathbf{w} - \mathbf{u} \odot \mathbf{b}\|^2 \\ \text{subject to} & \mathbf{w}^H \mathbf{G} \mathbf{w} \le \gamma \\ & \|\mathbf{u}_k\|^2 = 1, \ 1 \le k \le K \end{cases} \tag{40}$$

with $\mathbf{G} = \boldsymbol{\Sigma}_{\mathbf{c}}(\mathbf{X}_\star) + \mathbf{M}$. In the sequel, we propose a cyclic minimization to tackle the non-convex problem $\mathcal{P}_w^{synt}$.

For fixed $\mathbf{w}$, the problem $\mathcal{P}_w^{synt}$ boils down to the non-convex problem:

$$\begin{cases} \min\limits_{\mathbf{u}} & \left\| \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ . \\ . \\ \mathbf{a}_K \end{pmatrix} - \begin{pmatrix} \sqrt{g_1}\mathbf{u}_1 \\ \sqrt{g_2}\mathbf{u}_2 \\ . \\ . \\ \sqrt{g_K}\mathbf{u}_K \end{pmatrix} \right\|^2 \\ \text{subject to} & \|\mathbf{u}_k\|^2 = 1, \ 1 \le k \le K \end{cases} \tag{41}$$

where $\mathbf{a}_k$ contains the entries of $\mathbf{Aw}$ corresponding to $\mathbf{u}_k$ for $1 \leq k \leq K$. The above minimization can be decoupled into K minimization problems of the following form:

$$\begin{cases} \min_{\mathbf{u}_k} & \|\mathbf{a}_k - \sqrt{g_k}\mathbf{u}_k\|^2 \\ \text{subject to} & \|\mathbf{u}_k\|^2 = 1. \end{cases} \tag{42}$$

The solution to the above nearest-vector problem is simply given by

$$\mathbf{u}_k = \frac{\mathbf{a}_k}{\|\mathbf{a}_k\|}. \tag{43}$$

For fixed $\mathbf{u}$, the problem $\mathcal{P}_w^{synt}$ is equivalent to the following QCQP:

$$\begin{cases} \min_{\mathbf{w}} & \mathbf{w}^H \mathbf{A}^H \mathbf{A} \mathbf{w} - 2\Re\{(\mathbf{u} \odot \mathbf{b})^H \mathbf{A} \mathbf{w}\} \\ \text{subject to} & \mathbf{w}^H \mathbf{G} \mathbf{w} \leq \gamma. \end{cases} \tag{44}$$

Note that the positive definiteness of the matrices $\mathbf{A}^H \mathbf{A}$ and $\mathbf{G}$ ensures the convexity of the above QCQP. As a result, this QCQP can be solved efficiently via interior point methods or Lagrange multipliers [38].

• *Transmit sequence synthesis:* A technique similar to the above one can be used for transmit sequence synthesis. More precisely, we have

$$\mathbf{p}_k^H \left((\mathbf{xx}^H)^* \odot \mathbf{W}_\star\right) \mathbf{p}_k = \mathbf{x}^H \underbrace{\left((\mathbf{p}_k \mathbf{p}_k^H)^* \odot \mathbf{W}_\star\right)}_{\widetilde{\mathbf{T}}_k} \mathbf{x}. \tag{45}$$

Therefore, minimization of the following metric can be employed for transmit sequence synthesis:

$$\|\widetilde{\mathbf{A}}\mathbf{x} - \widetilde{\mathbf{u}} \odot \mathbf{b}\|^2 \tag{46}$$

where $\widetilde{\mathbf{T}}_k = \widetilde{\mathbf{V}}_k \widetilde{\mathbf{V}}_k^H$, $\widetilde{\mathbf{A}}^H = [\widetilde{\mathbf{V}}_1, \widetilde{\mathbf{V}}_2, \cdots, \widetilde{\mathbf{V}}_K]$ and $\widetilde{\mathbf{u}}, \mathbf{b}$ are defined similarly to the case of receive filter design ($\widetilde{\mathbf{u}}_k \in \mathbb{C}^{\widetilde{d}_k}$ with $\widetilde{d}_k$ being the rank of $\widetilde{\mathbf{V}}_k$). Note that for transmit sequence synthesis, the similarity and energy constraints should be taken into account in addition to the denominator of the $\widetilde{SINR}_{relax}(\nu)$. Consequently, we consider the following optimization problem to synthesize the sought transmit sequence:

$$\mathcal{P}_x^{synt} \begin{cases} \min_{\mathbf{x},\widetilde{\mathbf{u}}} & \|\widetilde{\mathbf{A}}\mathbf{x} - \widetilde{\mathbf{u}} \odot \mathbf{b}\|^2 \\ \text{subject to} & \mathbf{x}^H \boldsymbol{\Theta}_\mathbf{c}(\mathbf{W}_\star)\mathbf{x} \leq \zeta \\ & \|\mathbf{x}\|^2 \leq e \\ & \Re\{\mathbf{x}^H \mathbf{x}_0\} \geq \varepsilon \\ & \|\widetilde{\mathbf{u}}_k\|^2 = 1, \ 1 \leq k \leq K \end{cases} \tag{47}$$

where $\zeta = \text{tr}\{\mathbf{X}_\star \boldsymbol{\Theta}_\mathbf{c}(\mathbf{W}_\star)\}$ and $\varepsilon = e - \delta/2$. Let $\bar{\mathbf{x}}$ denote the optimal solution $\mathbf{x}$ to the above problem. Note that $\mathbf{x}_\star = \sqrt{e}\frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|}$ is such that $\|\mathbf{x}_\star\|^2 = e$, and $\Re\{\mathbf{x}_\star^H \mathbf{x}_0\} \geq \varepsilon$. Therefore, one can consider $\mathbf{x}_\star$ as

the optimized transmit sequence $\mathbf{x}_\star$ which lies in the desired similarity region and is feasible for the problem $\mathcal{P}$.

The non-convex optimization problem $\mathcal{P}_x^{synt}$ can be dealt with via a cyclic minimization similar to that used for $\mathcal{P}_w^{synt}$. For fixed $\mathbf{x}$, the solution to the kth resulting nearest-vector problem is given by

$$\widetilde{\mathbf{u}}_k = \frac{\widetilde{\mathbf{a}}_k}{\|\widetilde{\mathbf{a}}_k\|} \tag{48}$$

where $\widetilde{\mathbf{a}}_k$ includes the entries of $\widetilde{\mathbf{A}}\mathbf{x}$ corresponding to $\widetilde{\mathbf{u}}_k$ for $1 \leq k \leq K$. On the other hand, the case of fixed $\mathbf{u}$ is handled by solving the following convex QCQP:

$$\begin{cases} \min_{\mathbf{x}} & \mathbf{x}^H \left(\widetilde{\mathbf{A}}^H \widetilde{\mathbf{A}}\right) \mathbf{x} - 2\Re\{(\widetilde{\mathbf{u}} \odot \mathbf{b})^H \widetilde{\mathbf{A}} \mathbf{x}\} \\ \text{subject to} & \mathbf{x}^H \mathbf{\Theta}_\mathbf{c}(\mathbf{W}_\star)\mathbf{x} \leq \zeta \\ & \|\mathbf{x}\|^2 \leq e \\ & \Re\{\mathbf{x}^H \mathbf{x}_0\} \geq \varepsilon. \end{cases} \tag{49}$$

Remark 4: Note that the $\mathbf{x}_\star$ synthesized via the rank-one decomposition is a feasible point for the above convex QCQP. Indeed, the output of the rank-one decomposition procedure in Section IV-A can be used as a feasible starting point for the proposed cyclic minimization to obtain the transmit sequence. This can also be done for the receive filter synthesis. ∎

The steps of DESIDE can be summarized as in Table II. The first step consists of applying DESIDE-R to the relaxed problem $\mathcal{P}_1$ (see Table I). Steps 2 and 3 aim to synthesize high quality receive filters and transmit sequences, respectively. The cyclic minimizations in step 2 is terminated when a pre-defined stop criterion is satisfied; e.g. $\|\mathbf{w}^{(i+1)} - \mathbf{w}^{(i)}\| \leq \xi$ for a given $\xi > 0$ where i denotes the iteration number. A similar criterion can be used to terminate the algorithm in the step 3. Note that the obtained $\mathbf{x}$ after satisfying the stop criterion in the step 3 is scaled to obtain $\mathbf{x}_\star$ with energy e. The complexity of DESIDE can be addressed considering DESIDE-R and the synthesis stage. The complexity of each iteration of DESIDE-R is $\mathcal{O}(N^{3.5})$ (see the discussion above Table I). The complexity of each iteration of the proposed synthesis stage is determined by the complexity of solving the QCQPs in eqs. (44) and (49). These QCQPs can be solved via described methods in [30] with $\mathcal{O}(N^3)$ complexity.

V. NUMERICAL EXAMPLES

In this section we provide several numerical examples to examine the effectiveness of DESIDE method. Throughout the simulations, unless otherwise explicitly stated, we consider a code length[4] $N = 20$,

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[4]It is expected that the output SINR of the receive filter increases by increasing N due to the increase in the number of degrees of freedom for the design problem (see e.g. [39]) and the longer coherent processing interval [40].

TABLE II

THE DESIDE METHOD FOR OBTAINING DOPPLER ROBUST PAIR OF TRANSMIT SEQUENCE AND RECEIVE FILTER

Step 1 (Solving the relaxed problem): Apply DESIDE-R method to the optimization problem $\mathcal{P}_1$ to obtain the pair of $(\mathbf{W}_\star, \mathbf{X}_\star)$.

Step 2 (Receive filter synthesis): If $\mathbf{W}_\star$ is rank-one, perform an eigen-decomposition $\mathbf{W}_\star = \mathbf{w}_\star \mathbf{w}_\star^H$ to obtain $\mathbf{w}_\star$. Otherwise, initialize $\mathbf{w}$ with a random vector in $\mathbb{C}^N$ and do the following operations until a pre-defined stop criterion is satisfied:

- Obtain $\mathbf{u}$ by solving the optimization problem in (41) using (43).
- Solve the convex QCQP in (44) to obtain $\mathbf{w}$.

Step 3 (Transmit sequence synthesis): If $\mathbf{X}_\star$ is rank-one, perform an eigen-decomposition $\mathbf{X}_\star = \mathbf{x}\mathbf{x}^H$ to obtain $\mathbf{x}_\star = \mathbf{x}e^{j\arg(\mathbf{x}^H\mathbf{x}_0)}$. Otherwise, initialize $\mathbf{x}$ with a random vector in $\mathbb{C}^N$ and do the following operations until a pre-defined stop criterion is satisfied:

- Obtain $\tilde{\mathbf{u}}$ by solving the optimization problem in (47) for fixed $\mathbf{x}$ using (48).
- Solve the convex QCQP in (49) to obtain $\mathbf{x}$.

number of interfering range rings $N_c = 2$, and number azimuth sectors $L = 100$. The interfering signals backscattered from various azimuth sectors are weighted according to the azimuth beam-pattern characteristic of a typical linear array (see [12] for details). A uniformly distributed clutter is assumed with $\sigma_{(k,i)}^2 = \sigma^2 = 100$ for all (k,i). In addition, we let the Doppler shifts of the clutter scatterers be uniformly distributed over the interval $\Omega_c = [\bar{\nu}_d - \frac{\epsilon}{2}, \bar{\nu}_d + \frac{\epsilon}{2}] = [-0.1, 0.1]$ [40]. As to the target, we set $\alpha_T = 1$. Concerning the covariance matrix $\mathbf{M}$ of the signal-independent interference, it is assumed that $\mathbf{M}_{m,n} = \rho^{|m-n|}$ with parameter ρ. Regarding the similarity constraint, the generalized Barker code is used for sequence $\mathbf{x}_0$ [41]. This is a constant modulus sequence which has good correlation properties [12]. The size of the similarity region is controlled by $\delta_0 = \delta/e$. The total transmit energy e is supposed to be equal to the sequence length N. The convex optimization problems are solved via the CVX toolbox [42].

A. The effect of the design parameters

• *The width of Ω and the correlations between the interference samples*

The performance of the system generally depends on the width of target Doppler shift interval Ω and the correlations between the interference samples (controlled by the parameter ρ). Herein the non-robust design (i.e., with a priori known target Doppler shift $\tilde{\nu}$) of the transmit sequence and receive filter (with a similarity constraint) [12], i.e. the solution to the following problem:

$$\begin{cases} \underset{\mathbf{x},\mathbf{w}}{\max} \; SINR(\tilde{\nu}) \\ \text{subject to} \qquad \|\mathbf{x}\|^2 = e \\ \qquad\qquad \|\mathbf{x} - \mathbf{x}_0\|^2 \leq \delta \end{cases} \qquad (50)$$

is considered as a benchmark for comparisons. The effects of the width of interval Ω and the value of ρ are investigated in Fig. 1, where the values of $SINR(\nu)$ obtained by DESIDE (with $\mu = 10^{-3}$ and $\delta_0 = 0.5$) are compared with those of the non-robust design for two intervals $\Omega = [1, 3]$, $\Omega = [1.5, 2.5]$ and for $\rho \in \{0, 0.2, 0.5\}$. For the non-robust design, we reasonably set $\tilde{\nu}$ equal to $\frac{\nu_l + \nu_u}{2}$ with $\Omega = [\nu_l, \nu_u]$. In all examples, it is observed that DESIDE provides a robust $SINR(\nu)$ over the considered interval Ω of target Doppler shifts. The minimum value of $SINR(\nu)$ obtained by DESIDE outperforms that of the non-robust design significantly. The superiority of DESIDE is highlighted by observing that for a considerable range of the target Doppler shift ν, the $SINR(\nu)$ obtained by DESIDE is around 10 dB larger than that of the non-robust design. Furthermore, for any fixed ρ, the minimum value of $SINR(\nu)$ in the interval $\Omega = [1, 3]$ is less than that for $\Omega = [1.5, 2.5]$. As expected, the wider range of target Doppler shift leads to a more restricted design. Another observation is that for a fixed interval Ω, the minimum values of $SINR(\nu)$ increase as ρ increases. The observation is compatible with the behavior of the upper bound on the $\min_{\nu \in \Omega} \widetilde{SINR}_{relax}(\nu)$ in (29)- by increasing the value of ρ, the value of $\lambda_{min}(\mathbf{M})$ decreases and the upper bound on the $\widetilde{SINR}_{relax}(\nu)$ becomes larger. Note that in these examples, the ranks of the optimal $\mathbf{W}_\star$ and $\mathbf{X}_\star$ were equal to one (see section V-D) and hence the obtained pairs of the transmit sequence and the receive filter are *optimal* for the problem $\mathcal{P}$.

• *Size of the similarity region*

Examples for the robust design of transmit sequences and receive filters with various sizes of similarity region are now provided. The values of $SINR(\nu)$ obtained by DESIDE for different δ_0 in $\{0.01, 0.2, 0.4, 0.8\}$ are depicted in Fig. 2. The robustness property with respect to the target Doppler shift ν is observed in all examples. As expected, the larger the δ_0, the larger the worst value of the $SINR(\nu)$. This is due to a larger feasibility set for the optimization problem $\mathcal{SDP}_X$ and the fact that

the optimal $\mathbf{W}_\star$ and $\mathbf{X}_\star$ are rank-one.

B. Convergence of DESIDE-R

Examples of the convergence of DESIDE-R are depicted in Fig. 3. This figure shows the values of the objective function (in the maximization problem $\mathcal{P}_1$) obtained through the iterations $\kappa \in \mathbb{N}$ (with κ denoting the iteration number) for $\rho = 0.2$, $\delta_0 = 0.5$, as well as two intervals $\Omega = [1,3]$ and $\Omega = [1.5, 2.5]$. As expected, the cyclic maximization approach which is devised to tackle $\mathcal{P}_1$ leads to a monotonically increasing objective function $\min_{\nu \in \Omega} \widetilde{SINR}_{relax}(\nu)$. The values of the objective function for $\Omega = [1.5, 2.5]$ are larger than those for $\Omega = [1,3]$ (see the discussions associated with Fig. 1). Note that both $\mathbf{W}_\star$ and $\mathbf{X}_\star$ are rank-one here, and as a result, the obtained pairs of the transmit sequence and receive filter are *optimal* for the original design problem $\mathcal{P}$.

C. A fast-time coding example

As mentioned earlier (see Remark 1), the problem formulation and the design method can also be applied to fast-time coding systems. We present an example of such an application by considering $N = 32$ and $N_c = N$. The target Doppler shift ν is assumed to be in the interval $\Omega = [-0.1, 0.1]$. The considered maximum target Doppler shift corresponds to a target with an approximate velocity of Mach 4 illuminating by an L-band radar of sampling frequency 1 MHz. Owing to the fact that normalized Doppler shift in this case is proportional to the system bandwidth, we neglect the effect of the Doppler shifts of clutter scatterers. Fig. 4 shows the obtained $SINR(\nu)$ by DESIDE as well as the results for the non-robust design, for $\sigma^2 = 10$, $\delta_0 = 1$, and $\rho = 0$. It is observed that employing DESIDE leads to performance robustness of the system. In this example, the result obtained by DESIDE outperforms that of the non-robust design for $|\nu| \geq 0.035$. Moreover, the obtained $\mathbf{W}_\star$ and $\mathbf{X}_\star$ were rank-one too, similar to the examples presented earlier.

D. The synthesis algorithms

The performance analysis of the synthesis algorithms is performed by considering cases where the ranks of the solutions to the relaxed problem $\mathcal{P}_1$ are larger than one. We consider 20 random starting points for the synthesis algorithms (with $\xi = 10^{-3}$) and report the best result. In the first example, we assume $\Omega = [1,2]$, $\Omega_c = [-0.25, 0.25]$, $\sigma^2 = 100$, $\delta_0 = 0.1$. For a random initialization, DESIDE-R provides $(\mathbf{W}_\star, \mathbf{X}_\star)$ with $\text{rank}(\mathbf{W}_\star) = 2$ and $\text{rank}(\mathbf{X}_\star) = 1$ (it was numerically observed that as long as $\Omega \cap \Omega_c = \varnothing$, the rank of W_* is equal to one for most of the employed random initial points). The

optimal ESD corresponding to the pair $(\mathbf{W}_\star, \mathbf{X}_\star)$ is shown in Fig. 5(a). The values of $SINR(\nu)$ for the synthesized $\mathbf{w}_\star$ and $\mathbf{x}_\star$ are shown in Fig. 5(b). This figure also includes the optimal $\widetilde{SINR}_{relax}(\nu)$ (corresponding to $(\mathbf{W}_\star, \mathbf{X}_\star)$) and the result of applying the rank-one decomposition method. For the latter method, the best result is obtained with $\nu_\star = 1.71$, $\nu' = 1.3$, and $\nu'' = 1.5$. It is observed that using the proposed synthesis algorithm leads to values of $SINR(\nu)$ that are close to the optimal ones. Fig. 5(c) shows the optimal $\widetilde{SINR}_{relax}(\nu)$, $SINR(\nu)$ for $(\mathbf{w}_\star, \mathbf{x}_\star)$ synthesized via the proposed algorithm and the result of rank-one decomposition method for another case in which $\Omega = [1, 3]$. The performance of the rank-one decomposition method is degraded considerably whereas the difference between the results of the proposed algorithm and $\widetilde{SINR}_{relax}(\nu)$ is minor. This can be explained by noting that the rank-one decomposition method can consider the values of the optimal $\widetilde{SINR}_{relax}(\nu)$ at up to three points, i.e., $\nu_\star, \nu'$, and ν''. On the other hand, the proposed method considers a constrained synthesis problem to approximate the values of optimal $\widetilde{SINR}_{relax}(\nu)$ for an arbitrary set of discrete ν. To measure the goodness of the synthesis algorithms, we define the following loss metric:

$$\mathcal{L} \triangleq 10 \log \left(\frac{\min_{\nu \in \Omega} SINR(\nu)}{\min_{\nu \in \Omega} \widetilde{SINR}_{relax}(\nu)} \right). \tag{51}$$

In this example, the loss metric $\mathcal{L}$ for the proposed method and the rank-one decomposition method are equal to -0.25 dB and -4.1 dB, respectively. Next we study the effect of the number of optimal ESD samples, i.e. K, on the performance of the proposed synthesis stage. The results for a transmit sequence synthesis example are illustrated in Fig. 5(d). For this example, we have $\Omega = [0, 2], \Omega_c = [-0.125, 0.125]$, and $\delta_0 = 0.3$. Note that it was numerically observed that the rank of $\mathbf{X}_\star$ is equal to one as long as $\Omega \cap \Omega_c = \varnothing$. The figure shows the absolute values of loss metric $\mathcal{L}$ versus K. It is seen that the performance improvement for $K \geq 50$ is negligible. Another observation is that there is about -2 dB loss even for sufficiently large values of K. This might be due to imposing more constraints in the sequence synthesis as compared to the case of filter synthesis. In the example of Fig. 5(d), the loss of the rank-decomposition method is around -13 dB; here the latter method can take into account just one point of the optimal $\widetilde{SINR}_{relax}(\nu)$, i.e., $\nu_\star$.

VI. Concluding Remarks

A joint robust design of the transmit sequence and receive filter was considered for cases where the Doppler shift of the target is unknown. A novel method (called DESIDE) was proposed to tackle the design problem under the similarity constraint. The main results can be summarized as follows:

- The robust design problem was cast as a max-min problem by using the model which considers the effects of the interfering clutter scatterers at various range-azimuth bins and internal Doppler shifts of these scatterers. It was shown that for a given optimal transmit sequence, the problem can equivalently be written as a QCQP with infinitely many non-convex constraints and hence the design problem in general belongs to a class of NP-hard problems.

- DESIDE was devised to tackle the design problem. The method consists of solving a relaxed version of the design problem (via DESIDE-R) as well as a synthesis stage:

  - DESIDE-R was based on a reformulation of $SINR(\nu)$ by considering $\mathbf{W} = \mathbf{w}\mathbf{w}^H$ and $\mathbf{X} = \mathbf{x}\mathbf{x}^H$, relaxation of the rank-one constraints on the aforementioned matrices, and cyclic maximization of the relaxed problem. For fixed receive filter, the relaxed optimization problem was equivalently expressed as an SDP by using a transformation (inspired by Charnes-Cooper transform) and an SDP representation of the infinitely many affine constraints. Using a similar technique, an SDP was obtained in the fixed transmit sequence case.

  - New algorithms were devised to synthesize the receive filters and transmit sequences from the solutions to the relaxed problem. The synthesis algorithms aim to fit the $\widetilde{SINR}_{relax}(\nu)$ values associated with the solutions provided by DESIDE-R. The synthesis stage is cast as constrained non-convex problems which were dealt with via cyclic minimization.

- The effectiveness of the devised methods was illustrated by providing several numerical examples. It was shown that the DESIDE system performance possesses a considerable robustness with respect to the target Doppler shift. The numerical analysis of the proposed synthesis algorithms confirms that high quality pairs of receive filter and transmit sequence can be synthesized from the solutions to the relaxed problem.

The design problem considered in this paper is based on known parameters of clutter and signal-independent interference. Robust design of transmit sequences and receive filters with respect to uncertainties in clutter and interference parameters in addition to the target Doppler shift is a possible topic for future research.

APPENDIX A

PROOF OF LEMMA 1

Note that the numerator of $SINR(\nu)$ in (2) can be rewritten as

$$
\begin{aligned}
|\alpha_T|^2 \big| \mathbf{w}^H \left(\mathbf{x} \odot \mathbf{p}(\nu) \right) \big|^2 &= |\alpha_T|^2 \mathbf{w}^H (\mathbf{x} \odot \mathbf{p}(\nu))(\mathbf{x} \odot \mathbf{p}(\nu))^H \mathbf{w} \\
&= |\alpha_T|^2 (\mathbf{x} \odot \mathbf{p}(\nu))^H \mathbf{w}\mathbf{w}^H (\mathbf{x} \odot \mathbf{p}(\nu)) \\
&= |\alpha_T|^2 \mathbf{p}(\nu)^H \left(\mathbf{w}\mathbf{w}^H \odot \left(\mathbf{x}\mathbf{x}^H \right)^* \right) \mathbf{p}(\nu)
\end{aligned} \tag{52}
$$

where in the last equality we have used standard properties of the Hadamard product [31]. As to the denominator of $SINR(\nu)$ in (2), it is straightforward to verify that, for all (k,i),

$$
\begin{aligned}
\boldsymbol{\Gamma}(\mathbf{x},(k,i)) &= \mathrm{Diag}(\mathbf{x}) \boldsymbol{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d(k,i)}} \mathrm{Diag}(\mathbf{x})^H \\
&= \mathbf{x}\mathbf{x}^H \odot \boldsymbol{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d(k,i)}}.
\end{aligned} \tag{53}
$$

Using the matrix variable $\mathbf{X} = \mathbf{x}\mathbf{x}^H$ and substituting the above identity in (3) we obtain that

$$
\boldsymbol{\Sigma}_{\mathbf{c}}(\mathbf{X}) \triangleq \boldsymbol{\Sigma}_{\mathbf{c}}(\mathbf{x}) = \sum_{k=0}^{N_c-1} \sum_{i=0}^{L-1} \sigma_{(k,i)}^2 \mathbf{J}_k \left(\mathbf{X} \odot \boldsymbol{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d(k,i)}} \right) \mathbf{J}_k^T. \tag{54}
$$

As a result, eq. (52) and (54) yield the expression of $SINR(\nu)$ in (9).

To derive the alternative expression of $SINR(\nu)$ in (10), we begin by considering the result of the Lemma 3.1 in [12] which implies

$$
\mathbf{w}^H \boldsymbol{\Sigma}_{\mathbf{c}}(\mathbf{x}) \mathbf{w} = \sum_{k=0}^{N_c} \sum_{k=0}^{L-1} \sigma_{(k,i)}^2 \mathbf{x}^T \mathrm{Diag}(\mathbf{J}_{-k}\mathbf{w}^*) \boldsymbol{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d(k,i)}} \mathrm{Diag}(\mathbf{J}_{-k}\mathbf{w})\mathbf{x}^*. \tag{55}
$$

Note also that

$$
\mathrm{Diag}(\mathbf{J}_{-k}\mathbf{w}^*) \boldsymbol{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d(k,i)}} \mathrm{Diag}(\mathbf{J}_{-k}\mathbf{w}) = (\mathbf{J}_{-k}\mathbf{w}^*\mathbf{w}^T\mathbf{J}_{-k}^T) \odot \boldsymbol{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d(k,i)}}, \ \forall k. \tag{56}
$$

Therefore, using (56) as well as the fact that the covariance matrix $\boldsymbol{\Sigma}_{\mathbf{c}}(\mathbf{x}) \succeq \mathbf{0}$, we can write

$$
\mathbf{w}^H \boldsymbol{\Sigma}_{\mathbf{c}}(\mathbf{x}) \mathbf{w} = \mathbf{x}^H \boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{W}) \mathbf{x} \tag{57}
$$

where $\mathbf{W} = \mathbf{w}\mathbf{w}^H$ and

$$
\boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{W}) = \sum_{k=0}^{N_c-1} \sum_{i=0}^{L-1} \sigma_{(k,i)}^2 \left(\left(\mathbf{J}_k^T \mathbf{W} \mathbf{J}_k \right) \odot \left(\boldsymbol{\Phi}_{\epsilon_{(k,i)}}^{\bar{\nu}_{d(k,i)}} \right)^* \right). \tag{58}
$$

Now let $\beta = \mathbf{w}^H \mathbf{M}\mathbf{w}$, and observe that

$$
\mathbf{w}^H \boldsymbol{\Sigma}_{\mathbf{c}}(\mathbf{x}) \mathbf{w} + \mathbf{w}^H \mathbf{M}\mathbf{w} = \mathrm{tr}\left\{ \left(\boldsymbol{\Theta}_{\mathbf{c}}(\mathbf{W}) + \left(\frac{\beta}{e} \right) \mathbf{I} \right) \mathbf{X} \right\}. \tag{59}
$$

The above identity and eq. (52) prove the validity of the alternative expression of $SINR(\nu)$ in (10). ∎

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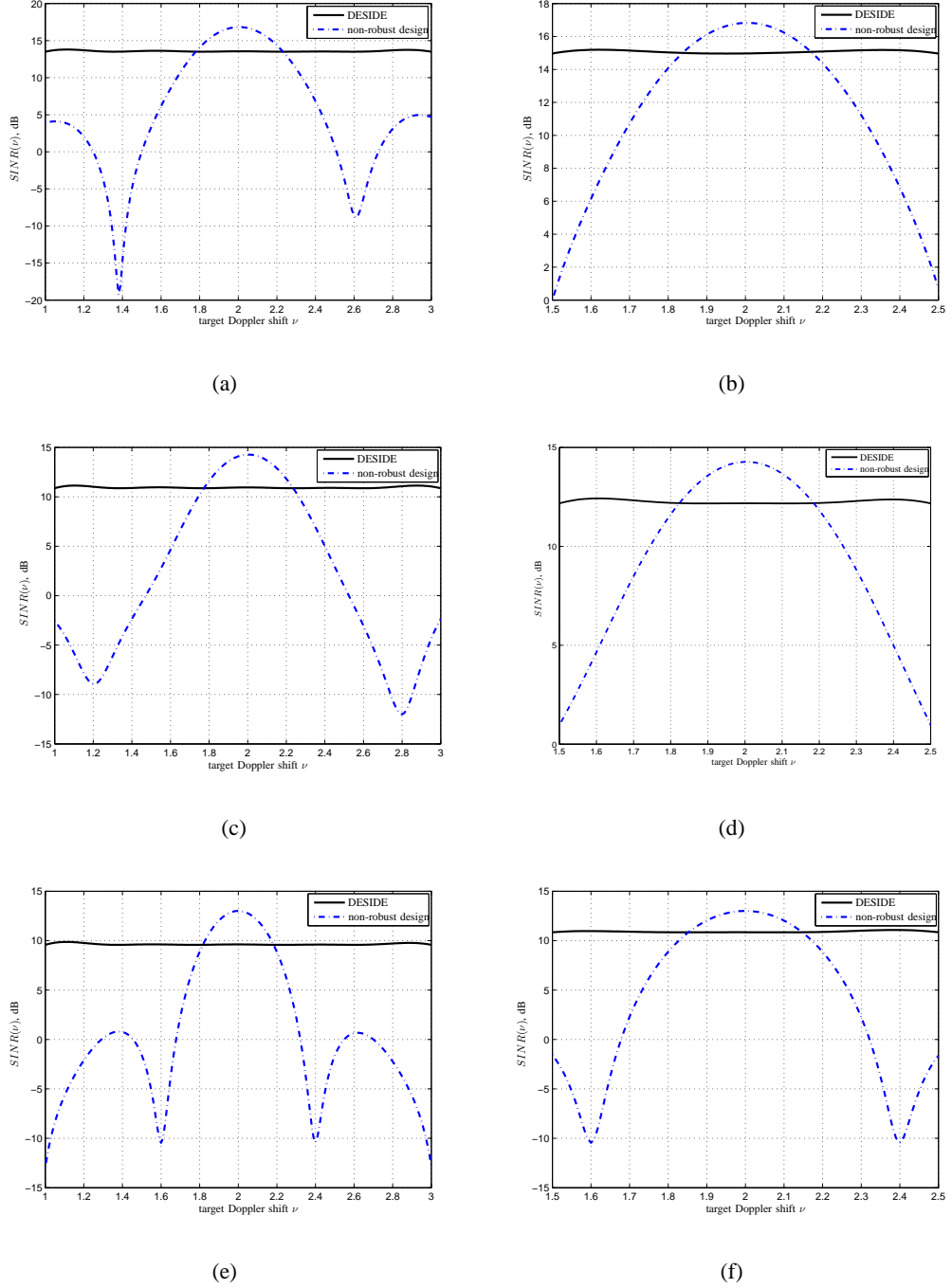[42] M. Grant and S. Boyd. (2012, February) CVX package. [Online]. Available: http://www.cvxr.com/cvx

Fig. 1. Design examples for various target Doppler shift intervals Ω and various ρ: (a) $\Omega = [1, 3]$ and $\rho = 0.5$, (b) $\Omega = [1.5, 2.5]$ and $\rho = 0.5$, (c) $\Omega = [1, 3]$ and $\rho = 0.2$, (d) $\Omega = [1.5, 2.5]$ and $\rho = 0.2$, (e) $\Omega = [1, 3]$ and $\rho = 0$, (f) $\Omega = [1.5, 2.5]$ and $\rho = 0$.
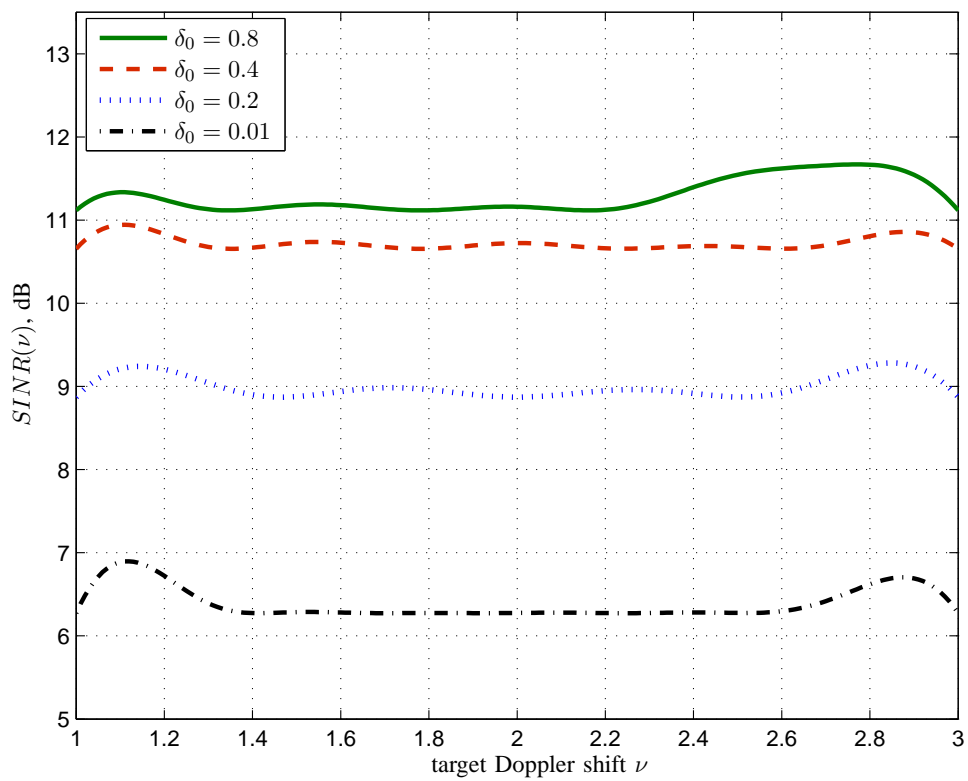
Fig. 2. Design examples for various sizes of the similarity region: $SINR(\nu)$ obtained by DESIDE versus target Doppler shift ν for $\delta_0 = 0.01, 0.2, 0.4,$ and 0.8.
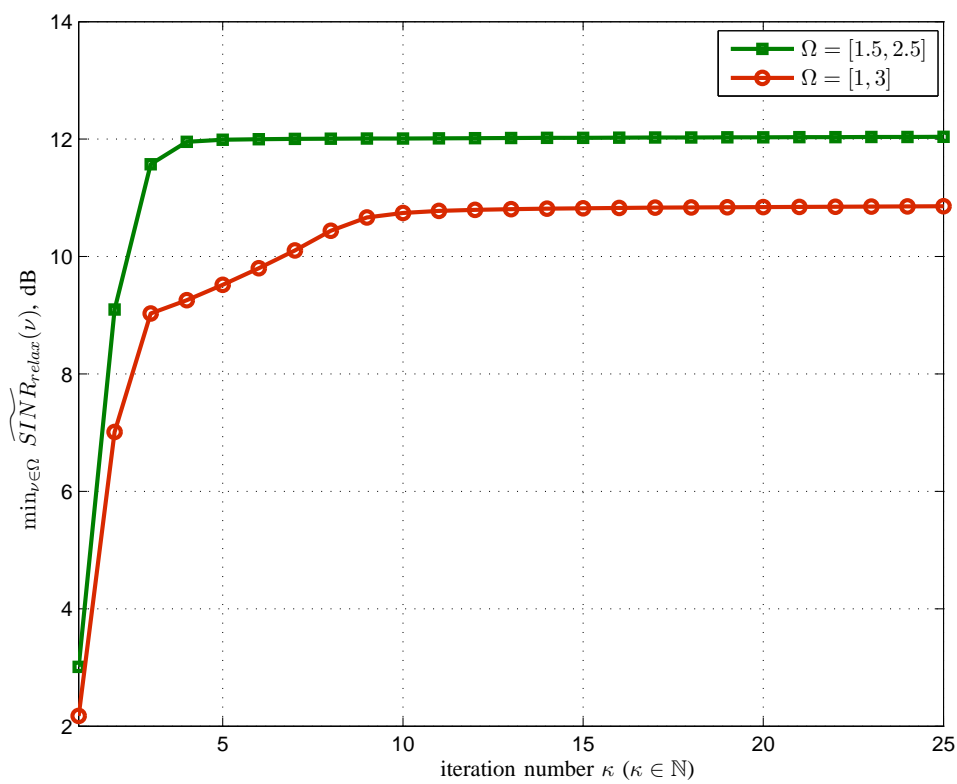
Fig. 3. The values of $\min_{\nu \in \Omega} \widetilde{SINR}_{relax}(\nu)$ obtained at different iterations of DESIDE-R for $\rho = 0.2$, $\delta = 0.5$ as well as two intervals $\Omega = [1, 3]$ and $\Omega = [1.5, 2.5]$.
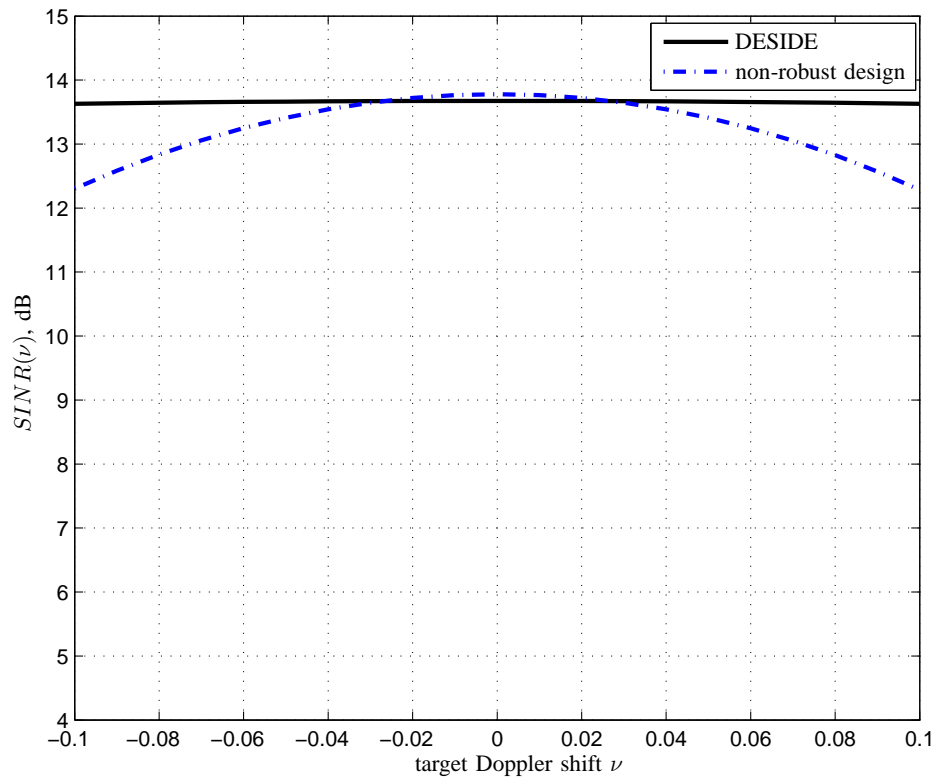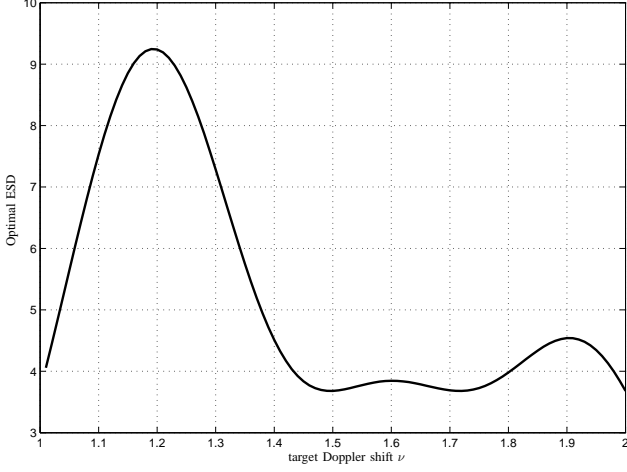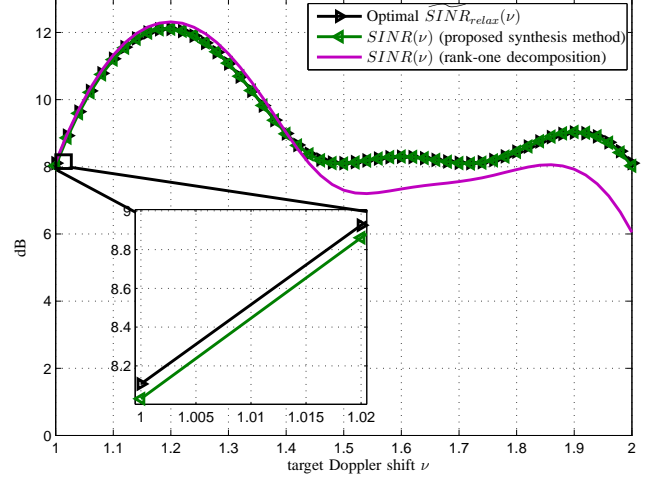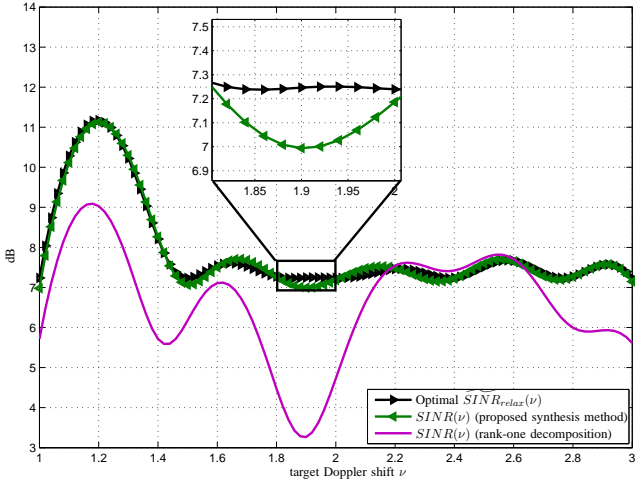
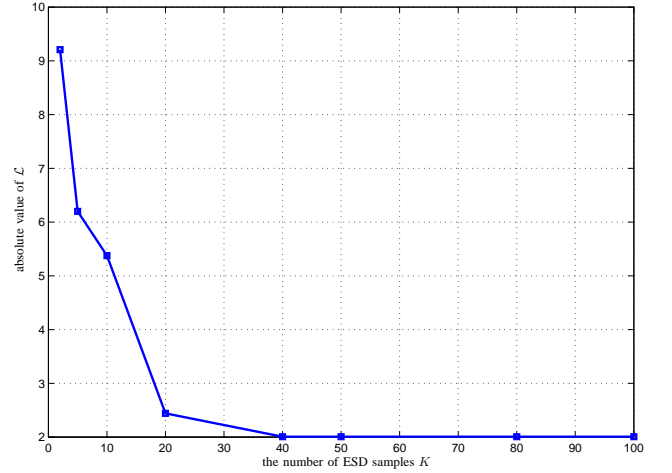Fig. 4.   Design example for a fast-time coding system.

(a)

(b)

(c)

(d)

Fig. 5.   Results obtained with the proposed synthesis algorithms: (a) an optimal ESD, (b) results of the filter synthesis corresponding to part (a), (c) another filter synthesis example, (d) absolute value of the loss metric $\mathcal{L}$ of the transmit sequence synthesis versus the number of ESD samples K for the considered transmit sequence synthesis example. The zoomed areas in (b) and (c) show the values of the optimal $\widetilde{SINR}_{relax}(\nu)$ and $SINR(\nu)$ of the proposed synthesis method in the neighborhoods of their minimum values.

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