Exact theoretical performance analysis of optimum detector in statistical multi-input multi-output radars

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Abstract: This study is concerned with the performance analysis of detection problem in statistical multiple-input multiple-output radars for Gaussian interference. This subject has been addressed in some publications for such special cases as white Gaussian noise and orthogonal transmission. However, theoretical performance analysis of optimum detector for general case including coloured Gaussian interference and arbitrary transmission signal has not been reported yet. In the present study, after developing the optimum detector for a general case, exact closed-form expressions are derived for the probability of detection and false alarm. As the derived expressions have complicated form, their interpretation is not tractable in the general case. Therefore lower and upper Chernoff bounds are obtained to provide better insight into the detector performance. Furthermore, the effect of various parameters on the detector performance is investigated by Monte-Carlo simulations. Numerical analysis shows a high degree of consistency between the theoretical and Monte-Carlo simulation results.

1 Introduction

Multiple-input multiple-output (MIMO) radar is a relatively new concept originating from its variation in control and communication theory. In general, these systems include two main groups [1]. The first is known as statistical MIMO radar or MIMO radar with widely separated antennas introduced in [2, 3]. In this architecture, transmitters and/or receivers are distributed spatially and hence there is a considerable distance between them as compared with the wavelength of the radar signal. Another type of MIMO radar uses collocated architecture, like phased-array radars, in which various transmission nodes transmit arbitrary waveforms, as opposed to phased-array systems [1, 4]. Collocated MIMO radars’ literatures mainly discuss potential advantages of these systems due to the higher degrees of freedom provided by waveform diversity [1, 4]. These advantages are parameter identifiability, clutter migration, spatial power distribution, and so on.

Spatial distribution in statistical MIMO radars provides angular diversity that can improve detection and estimation performance of these systems [2, 3, 5]. In fact, existence of different aspect angles from transmitters to the target and from the target to the receivers can be used to reduce the effect of the target radar cross-section (RCS) fluctuations on system performance. In other words, target RCS fluctuation in radar applications plays the role of fading in communication systems. Therefore like common diversity methods used in communication systems to decrease the probability of deep fading, angular diversity can be used in radar applications to provide better performance. It is worth noting that although one may find some similarities between statistical MIMO radars and multi-static ones, the former is a generalised version of the latter, since the MIMO radars benefit from joint processing as opposed to multi-static ones using decentralised processing scheme [1, 6, 7]. One of the major advantages of angular diversity is detection performance improvement as many references have been considered in this issue. In the following, a review of the most important studies regarding target detection in Gaussian disturbance is given. In [2], a main reference, detection performance of statistical MIMO radar has been developed for Swerling-I target model in white noise for single-pulse detection and orthogonal transmission. In fact, it assumes a white Gaussian distribution for target (complex) amplitude. Optimum detector, a variation in energy detector, has been derived. Also, its performance has been evaluated and compared with a phased-array system in order to better illustrate the improving effect of the angular diversity. Finally, [2] proposes a detector for the case in which the noise variance is unknown. Theoretical expressions derived for detection performance in [8] are for conditions similar to those in [2], which especially include white noise and orthogonal transmission; however, a generalisation of the target modelling (finite scattering) has been used in comparison with [2]. Another reference highly related to [2] is [9] in which the KL-divergence is used as a metric for performance analysis in white noise and orthogonal transmission. In [10], detection performance of optimum detector for orthogonal transmission and white noise has been simulated using both Swerling-I and Swerling-II target models.
In [11], optimum detector in the white Gaussian noise (with known variance) has been developed under various target amplitude models for single-pulse detection and arbitrary transmission. Also, an integral form for multi-pulse detection has been introduced. In addition, probability density function (PDF) of the test statistic with an implicit form for involved coefficients has been provided for Swerling-I and II in white disturbance. Note that in [11] it has been assumed that several samples were taken from each echo.

Coloured interference which is more compatible with real situations concerning clutter background has been considered in some works like [12]. In fact, in [12] optimum detector has been developed for multi-pulse detection, a white Gaussian target, coloured Gaussian clutter and arbitrary transmission. Also, performance analysis has been performed using deflection coefficient related to the first- and second-order moments of the test statistic.

It has been assumed in several papers that target amplitude is an unknown deterministic parameter and hence they have pursued generalised likelihood-ratio test (GLRT) approach, a suboptimum but popular one. For example, in [13, 14] GLRT detector for coloured Gaussian clutter, multi-pulse detection and arbitrary transmission has been derived. Although conditional detection performance of that detector has been calculated, there is no closed-form expression for unconditional detection probability. Also, GLRT method has been used in adaptive detection procedure, for example, works of [7, 15]. In [7], for coloured Gaussian clutter with unknown covariance, a target with unknown deterministic amplitude and orthogonal transmission, two types of adaptive detectors have been developed. Note that performance analysis has been performed by simulation in this reference. In the work of [15], performance of adaptive detectors has been investigated using deflection (divergence) coefficient. Also, expressions for unconditional performance have been given in the form of some expected values over unknown parameters. In [16], for similar conditions like those of [7], diagonal loading has been proposed to improve the performance of covariance matrix estimation. Some special cases of the work in [7] have been repeated in [17]. Also, GLRT approach has been employed in [18] for handling the problem involving a white Gaussian target with unknown variance and coloured clutter with unknown covariance.

A few works assumed model-based clutter, for example, auto-regressive model has been used in [19], and GLRT approach has been employed for estimation of model’s unknown parameters. It is worth noting that in the cited works, a stationary target or a target with known Doppler shift has been assumed. Also, in all of them some variations of range-gating processing have been used. In contrast to these works, in [20] a novel method has been proposed which jointly estimates the delay vector (corresponding to different sites) and solves detection problem instead of using range-gating process. Underlying assumptions in [20] include white Gaussian noise, a white Gaussian target with known variances and orthogonal transmission.

As stated before, detection problem has been less important in collocated MIMO radars. Anyways, the ultimate goal of some references such as [21, 22] to maximise the signal-to-interference-plus-noise (SINR) at the output of the detector. They have assumed coloured Gaussian models with known covariance matrices for both clutter and target. Note that there is no analytic performance evaluation for detection problem in these references. Finally, we cite the work of [23] which has exploited iterative GLRT approach for multi-target detection under unknown assumption for both targets’ amplitude and clutter covariance.

In this paper we assume a general situation involving multi-pulse detection problem, a Swerling-I target, coloured Gaussian interference with known covariance and arbitrary transmission. Our main contribution is the derivation of the theoretical closed-form expressions (with explicit parameters) for probability of detection and probability of false alarm regarding the optimum detector developed for mentioned conditions.

This paper is organised as follows. In Section 2, problem modelling is introduced. The optimum detector for multi-pulse detection, arbitrary transmission and coloured interference is developed in Section 3. Section 4 contains derivation of the exact closed-form expression for detection performance. Chernoff bound is given in Section 5 to come to a better understanding of detector performance. In Section 6, some numerical results are provided to validate the theoretical derivations. Finally, summary and conclusion are provided in Section 7.

### 2 Problem Modelling

In the present study, we assume a narrowband statistical MIMO radar scenario containing \( N_r \) receiving nodes and \( N_t \) transmission nodes which are stationary and have been distributed over space such that there is angular diversity in the system, similar to the stated scenario in [1, 13, 14]. The only difference lies in the fact that in [1, 13, 14] it has been assumed that target amplitude is unknown deterministic whereas we assume Gaussian target. Also, assume that arbitrary transmitted waveform from transmitters can be expressed by the weighted sum of an \( N \)-dimensional orthonormal basis such as \( \{ \phi_n(t) \}_{n=1}^{N} \). In fact, transmitted waveform from the \( m \)th transmitter, \( s_m(t) \), is

\[
s_m(t) = \sum_{n=1}^{N} p_{n,m} \phi_n(t); \quad 0 \leq t \leq T_s; \quad m = 1, 2, \ldots, N_t
\]

(1)

where \( p_{n,m} \); \( n = 1, 2, \ldots, N \) are employed coefficients for generation of desired waveform at the \( m \)th transmission node, and \( T_s \) is the transmission time of the \( s_m(t) \). Therefore received signal at the \( \ell \)th receiver, \( r_{\ell}(t) \), becomes

\[
r_{\ell}(t) = \sum_{m=1}^{N_t} \sum_{n=1}^{N} p_{n,m} \phi_n(t - \tau_{m,\ell}) \alpha_{m,\ell} + c_{\ell}(t)
\]

(2)

where \( \alpha_{m,\ell} \) is the (complex) amplitude of the target and channel effects associated with the path from the \( m \)th transmitter to the \( \ell \)th receiver, \( \tau_{m,\ell} \) is the delay corresponding to that path, and \( c_{\ell}(t) \) is the interference (including noise and clutter) at the \( \ell \)th receiver. It can be shown that under narrowband assumption, \( \tau_{m,\ell} \) cannot be resolved from each other and can be expressed as \( \tau \) [1, 2, 13, 14]. Finally, after projection of received signals onto signal space [14], the binary hypothesis involving decision about the existence of a target at the \( \ell \)th receiving node
becomes [1, 13, 14]

\[
\begin{align*}
H_0: r_i &= c_i \\
H_1: r_i &= Pa_i + c_i
\end{align*}
\]  

(3)

where \( r_i \) is a vector containing received samples at \( i \)th receiving node, \( a_i \) is the form of \( a_{m,i} \) in the form of \( a_i = (a_{1,i}, \ldots, a_{N_r,i})^T \) and \( P \) is a known \( N \times N_i \) matrix with \( [P]_{m,n} = p_{m,n} \). Also, it is assumed that interference vector, \( c_i \), is a circularly symmetric Gaussian random process (CSGRP) with known covariance denoted by \( M_i \). In addition, the Swerling-I target model is used for target amplitude, a common assumption in statistical MIMO radar due to the reality of angular diversity [1, 2, 12]. In essence, \( a_i \) is modelled by a white CSGRP with variance \( \sigma_r^2 \).

It should be noted that the mentioned modelling is valid for a stationary target or a target with known Doppler shift [1, 13, 14]. Also, it is worth noting that a well-known special case for the above modelling arises for narrowband pulsed-radar where \( \phi_p(t) \equiv p(t - (n - 1)T_{PRI}) \), \( p(t) \) is a unit-energy pulse and \( T_{PRI} \) is the pulse repetition period of the radar [1, 13, 14]. Consequently, discrete samples in (3) are derived by sampling the matched-filtered version of the received signal at times equal to \( t + (k - 1)T_{PRI} \) for \( k = 1, 2, \ldots, N \).

As signal processing is performed jointly in MIMO radars, the binary hypothesis problem considering all received samples should be solved, so we have

\[
\begin{align*}
H_0: r &= c \\
H_1: r &= Ba + c
\end{align*}
\]  

(4)

where \( r, c, a \) and \( B \) are defined as

\[
\begin{align*}
r &= [r_1^T, r_2^T, \ldots, r_{N_r}^T]^T \\
c &= [c_1^T, c_2^T, \ldots, c_{N_r}^T]^T \\
a &= [a_1^T, a_2^T, \ldots, a_{N_r}^T]^T \\
B &= I_{N_r} \otimes P
\end{align*}
\]

where \( I_{N_r} \) is \( N_r \times N_r \) identity matrix and \( \otimes \) denotes the Kronecker product.

Owing to the fact that transmitters and receivers are widely separated, it is assumed that both \( c_i \)s and \( a_i \)s are uncorrelated random vectors for various \( i \) and hence they are independent [7, 13, 14, 16]. Indeed, one can write

\[
\begin{align*}
\text{cov}(a_n, a_m) &= \delta(n - m)\sigma_r^2 I_{N_r} \\
\text{cov}(c_n, c_m) &= \delta(n - m)M_n
\end{align*}
\]

(5a)

where \( \delta(n - m) \) is discrete-time delta function. Consequently, we have

\[
\begin{align*}
\mathcal{E}\{x_n^{H}H_0\} &= \mathcal{E}\{n_n^{H}\} = I_{N_r} \\
\mathcal{E}\{x_n^{H}H_1\} &= \mathcal{E}\{(s_i + n_i)(s_i + n_i)^{H}\} = \mathcal{E}\{M_i^{-1/2}Pa_is_i^{H}P_i^{H}M_i^{-1/2} + n_n^{H}\} = \sigma_r^2 M_i^{-1/2}PP_i^{H}M_i^{-1/2} + I_{N_r} \triangleq A_i + I_{N_r}
\end{align*}
\]  

(5b)

\[
\begin{align*}
\mathcal{E}\{nn^{H}\} &= I_{N_r} \\
\mathcal{E}\{SS^{H}\} &= \mathcal{E}\{R^{-1/2}Ba_d^{H}B^{H}R^{-1/2}\} = \sigma_r^2 R^{-1/2}BB^{H}R^{-1/2} \triangleq C_s
\end{align*}
\]  

(6b)

where \( R \) is a \( N_r \times N_r \) block-diagonal matrix defined as \( R \triangleq \text{blkdiag}(M_1, M_2, \ldots, M_{N_r}) \).

3 Detector derivation

In this section the optimum detector for (4) will be developed. To this end, whitened problem is formed via multiplication of (4) by \( R^{-1/2} \) as

\[
\begin{align*}
H_0: x &= n \\
H_1: x &= R^{-1/2}Ba + n
\end{align*}
\]  

(5a)

where \( x \triangleq R^{-1/2}r \) and \( n \triangleq R^{-1/2}c \). In other words, \( x_i \triangleq M_i^{-1/2}r_i \), \( n_i \triangleq M_i^{-1/2}c_i \), \( s_i \triangleq M_i^{-1/2}Pa_i \) and we have (see (5b))

Note that \( A_i \triangleq \sigma_r^2 M_i^{-1/2}PP_i^{H}M_i^{-1/2} \). By defining \( S \triangleq R^{-1/2}Ba \), one can write

\[
\begin{align*}
H_0: x &= n \\
H_1: x &= S + n
\end{align*}
\]  

(6a)

where (see (6b))

and also \( C_s = \text{blkdiag}(\sigma_r^2 M_1^{-1/2}PP_1^{H}M_1^{-1/2}, \ldots, \sigma_r^2 M_{N_r}^{-1/2}PP_{N_r}^{H}M_{N_r}^{-1/2}) = \text{blkdiag}(A_1, \ldots, A_{N_r}) \).

Therefore the likelihood-ratio for (6) becomes

\[
L(x) = \frac{f_s(x|H_1)}{f_s(x|H_0)} = \prod_{i=1}^{N_r} \frac{f_s(x_i|H_1)}{f_s(x_i|H_0)} = \prod_{i=1}^{N_r} \frac{\det (I + A_i)}{\det (I)}
\]

\[
\exp(-x_i^{H}(I + A_i)^{-1}x_i) \times \exp(-|x_i|^2)
\]

(7)

where the second equality holds because \( x_i \)s are independent for various \( i \). After some manipulations similar to the procedure employed in [24], one can write the test statistic as

\[
T(x) = \sum_{i=1}^{N_r} x_i^{H}A_i(I + A_i)^{-1}x_i \geq \eta
\]

(8)

Now assume that eigenvalue decomposition of \( (1/\sigma_r^2)A_i \) becomes \( \mathcal{V}_i \mathcal{A}_i \mathcal{V}_i^H \) where \( \mathcal{A}_i \) is a diagonal matrix containing eigenvalues, that is, \( \lambda_{ki} \) for \( k = 1, 2, \ldots, N_r \) and \( \mathcal{V}_i \) is a unitary matrix including column eigenvectors. Consequently, we have

\[
A_i(I + A_i)^{-1} = \mathcal{V}_i \sigma_r^2 \mathcal{A}_i (\sigma_r^2 \mathcal{A}_i + I)^{-1} \mathcal{V}_i^H
\]

(9)
By defining $y_i = V_i^H x_i$ and using (9), the test statistic in (8) can be written in its canonical form after some algebra as

$$T(x) = \sum_{i=1}^{N} \sum_{k=1}^{N} \left( \frac{\sigma^2_0 \lambda_{ijk}}{1 + \sigma^2_0 \lambda_{ijk}} \right) |y_{k,i}|^2 > \eta \quad (10)$$

It is worth noting that (10) is compatible with the result derived in [25] by assuming $M_f = M$, $V_i$ leading to $\lambda_{ijk} = \lambda_i$, $\forall i$. Also, some terms in (10) may be zero because the number of non-zero eigenvalues is equal to the rank of $A_i$, for $i = 1, 2, \ldots, N_r$, which is not necessarily a full-rank matrix.

## 4 Performance analysis

The main contribution of this paper is derivation of the exact closed-form expressions for both probability of detection and probability of false alarm for optimum MIMO radar detector for coloured Gaussian interference, Swerling-I target model, multi-pulse detection and arbitrary transmission waveforms.

In the following, closed-form expressions for performance analysis are derived in the general case first. After this, the reduced results in some special cases will be investigated.

### 4.1 General case

The PDF of $T(x)$ is required for performance evaluation of the optimum detector. For this purpose, we consider (10) and begin from the PDF of $y_i$. According to (6), for $x$, we have

$$\begin{align*}
H_0: x &\sim CN(0, I) \\
H_1: x &\sim CN(0, A_1 + I)
\end{align*}$$

Therefore the covariance of $y_i$ becomes (see equation at the bottom of the page) which results in

$$\begin{align*}
H_0: y_i &\sim CN(0, I) \\
H_1: y_i &\sim CN(0, A_i + \sigma_0^2 I)
\end{align*}$$

Consequently, elements of $y_i$, that is, $y_{k,i}$, $k = 1, 2, \ldots, N$ are independent because these uncorrelated random variables (RV) are jointly Gaussian RVs at both hypotheses. Accordingly

$$\begin{align*}
H_0: y_{k,i} &\sim CN(0, 1) \\
H_1: y_{k,i} &\sim CN(0, 1 + \sigma_0^2 \lambda_{ijk})
\end{align*}$$

After the mentioned manipulations, performance of optimum detector can be found using the following lemma proved in Appendix 1.

**Lemma 1**: If $z = \sum_{l=1}^{L} \sum_{m=1}^{Q} g_{l,m} h_{l,m}$ where $h_{l,m}$ are $x^2$-independent and identically distributed (i.i.d) RVs, and $\{g_{l,m}\}$ are positive values, then (see (14a))

where $\left\{h_{n,k}\right\}_{n=1}^{N} \sim \left\{A\right\}_{n=1}^{N}$ is a new sequence containing distinct values of $\{g_{l,m}\}$, $m_{n,k}$ is the multiplicity of $h_{n,k}$, $Y(s) = \prod_{l=1}^{L} \prod_{m=1}^{Q} (1 - 2s h_{n,k})^{-1}$, and $f_{x^2_{z}}(z)$ is the PDF of a $x^2_z$ RV. Also, the right-hand tail probability of $z$ becomes (see (14b))

where $\gamma(\cdot, \cdot)$ and $\Gamma(\cdot)$ are the lower incomplete gamma function and gamma function, respectively.

$$\begin{align*}
f_z(z) &= \sum_{n=1}^{n_z} \sum_{k=1}^{k_z} \sum_{j=1}^{j_z} D_{n,k,j} \times \frac{1}{b_{n,k}} \times f_x(z/b_{n,k}), \quad z \geq 0 \\
Q_z(z) &= \text{Prob}(Z \geq z) = \int_{z}^{\infty} f_z(x) dx = \sum_{n=1}^{n_z} \sum_{k=1}^{k_z} \sum_{j=1}^{j_z} D_{n,k,j} Q_{x^2_{z,j}}(z/b_{n,k})
\end{align*}$$

$$\begin{align*}
\frac{Q_{z}(z)}{\gamma(\cdot, \cdot)} &= \frac{1 - \gamma(j, z/2)}{\Gamma(j)} \sum_{n=0}^{j} \left(\frac{z/2}{n!}\right) \exp(-z/2)
\end{align*}$$
For employing stated lemma, set $g_{ij} = (\sigma_i^2 \lambda_{ij}) / 2$, $h_{ij} = z_{ij}$ and also define a new sequence containing distinct values of $\{\lambda_{ij}\}_{i=1,2,\ldots,N}$ which are denoted by $\{\lambda_{ij}^{(d)}\}_{i=1,2,\ldots,N}$ such that the multiplicity of $\lambda_{p,w}$ becomes $m_{p,w}$. Therefore by using (14b) the probability of detection, that is, $P_d$, for the test statistic in (13) can be written as the following (see (15))

$$
D_{p,w}^{(d)} \triangleq \frac{1}{(m_{p,w} - j)!} \left[ \frac{d^{m_{p,w}-j}}{d\gamma_{p,w}^{m_{p,w}-j}} (1 - s \gamma_{p,w})^{m_{p,w}-j} \right]_{\gamma = 1 / \tilde{b}_{p,w}}
$$

where

$$
\tilde{b}_{p,w} \triangleq \frac{\sigma_i^2 \lambda_{p,w}^{(d)}}{1 + \sigma_i^2 \lambda_{p,w}^{(d)}}
$$

$$
\tilde{\gamma}(s) \triangleq \prod_{i=1}^n \prod_{k=1}^p \left( \frac{1}{1 - s \tilde{b}_{i,k}} \right)^{m_{i,k}}
$$

Although in (15) and (16) we propose the exact closed-form expressions for $P_d$, $P_f$, for general case which could be efficiently calculated by computers, they are not tractable and so interpretation of these equations is difficult. Therefore in the following, we derive detection performance for some special cases.

### 4.2 Identical interference covariance

In this situation we assume that the covariance of the interference in all of the receivers is the same. In essence

$$
M_i \triangleq M, \quad \forall i \Rightarrow \lambda_{ij} \triangleq \lambda_k
$$

Therefore, (13) can be rewritten as

$$
\begin{align*}
H_0: \ T(x) &= \sum_{i=1}^N \sum_{k=1}^N \left( \frac{\sigma_i^2 \lambda_k}{1 + \sigma_i^2 \lambda_k} \right) \times \frac{1}{2} (z_{i,k}^0) \\
H_1: \ T(x) &= \sum_{i=1}^N \sum_{k=1}^N (\sigma_i^2 \lambda_k) \times \frac{1}{2} (z_{i,k}^1)
\end{align*}
$$

We can express $T(x)$ in (18) more compactly considering that $z_{i,k}^0$ are independent RVs for $i = 1, 2, \ldots, N$, because of the independency of $x_i$ for both $q = 0$ and 1. Since the sum of independent chi-square RVs is also a chi-square RV with degrees of freedom equal to the sum of degrees of freedom of individual ones [24], $T(x)$ can be expressed as (see (19))

$$
\begin{align*}
H_0: \ T(x) &= \sum_{k=1}^N \left( \frac{\sigma_i^2 \lambda_k}{1 + \sigma_i^2 \lambda_k} \right) \times \frac{1}{2} W_k^0 \\
H_1: \ T(x) &= \sum_{k=1}^N (\sigma_i^2 \lambda_k) \times \frac{1}{2} W_k^1
\end{align*}
$$

where $W_k^0$ and $W_k^1$ are independent $\chi^2_{2N}$ RVs for $k = 1, 2, \ldots, N$.

It can be seen from (19) that $T(x)$ in this special case is a weighted sum of chi-square RVs with $2N$ degrees of freedom. Therefore increasing the number of receivers, that is, $N$, can enlarge the degrees of freedom of these RVs and improve the overall performance as expected from the general case interpretation explained in Section 4.1. Again, analysis of the number of transmitters’ effect is more complicated because the rank of $M^{-1/2}PP^TM^{-1/2}$ dictates the number of non-zero $\lambda_k$ and hence the number of effective summing terms in (19); in fact

$$
\begin{align*}
H_0: \ T(x) &= \sum_{k=1}^N \left( \frac{\sigma_i^2 \lambda_k}{1 + \sigma_i^2 \lambda_k} \right) \times \frac{1}{2} W_k^0 \\
H_1: \ T(x) &= \sum_{k=1}^N (\sigma_i^2 \lambda_k) \times \frac{1}{2} W_k^1
\end{align*}
$$

where $\Delta_1 \triangleq \text{rank}(M^{-1/2}PP^TM^{-1/2}) \leq \Delta \triangleq \min(N, 1)

Probability of detection, $P_d$, for optimum detector in (19) is obtained by corollary of Lemma 1 proved in Appendix 2.

**Corollary 1:** If $c = \sum l_i a_i y_i$ where $y_i$ are $\chi^2_1$, i.d RVs, and $a_i$ are positive values such that the multiplicity of $a_i$ is $u_i$, then

$$
f_c(c) = \sum_{i=1}^{l_i} \sum_{k=1}^{u_i} A_{i,k} \times \frac{1}{a_i} \frac{1}{(n_i - k)!} \frac{1}{\text{det}(1 - 2a_i)^{n_i} Y(s)}
$$

where $l_i$ is the number of distinct $a_i$ such that

$$
Y(s) \triangleq \prod_{i=1}^{l_i} \left( \frac{1}{1 - 2a_i} \right)^{u_i}
$$

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Also, the right-hand tail probability of \( c \) becomes

\[
Q_c(c) = \sum_{i=1}^{L} \sum_{k=1}^{uL} A_{i,k} Q_{X_{2a}}(c/a_i)
\]

where \( Q_{X_{2a}}(c) = 1 - \frac{\gamma(k, c/2)}{\Gamma(k)} \sum_{n=0}^{k-1} \frac{(c/2)^n}{n!} \exp(-c/2) \)

(21b)

Using Corollary 1, one can express the probability of detection and the probability of false alarm by setting \( f = N \) and proper replacement of \( a_i \) and \( b_i \) in each hypothesis as the following (e.g. in \( H_1 \) we have

\[
a_i = (\sigma^2_i \lambda_k)/2, \quad b_i = W_i^2
\]

and

\[
\begin{align*}
    P_d &= \sum_{k=1}^{N_x} \sum_{j=1}^{N_x} D_{kj} Q_{X_{2a}}((2 \eta/\sigma^2_j \lambda_k) \\
    P_{fa} &= \sum_{k=1}^{N_x} \sum_{j=1}^{N_x} \hat{D}_{kj} Q_{X_{2a}}((2 \eta(1 + \sigma^2_j \lambda_k)/\sigma^2_j \lambda_k)
\end{align*}
\]

(22a)

where (see (22b))

Also \( N_x \) is the number of distinct \( \lambda_k \)s, \( n_k \) is the multiplicity of \( \lambda_k \)s such that \( (\sum n_k = N) \), and \( D_{kj}, \hat{D}_{kj} \) have been found by proper replacement in (21a). In the following, performance of the optimum detector is analysed in more special situations.

4.2.1 Multiple-input single-output (MISO) radars: In this case, any configuration for transmitted waveforms is possible, \( T(x) \) is the weighted sum of \( X_2 \) RVs according to (19) at both hypotheses. Therefore substituting \( N_x = 1 \) in (22a), using (22b) and also assuming distinct \( \lambda_k \)s \((N_x = N, n_k = 1; \forall k)\) for the detection performance, we have

\[
\begin{align*}
    \hat{D}_{kj} &= \frac{1}{(n_k N_x - j)!} \left[ \frac{d^{n_k N_x - j}}{dx^{n_k N_x - j}} (1 - s \sigma^2_j \lambda_k)^{y_k N_x} \times \prod_{k=1}^{N_x} \left( \frac{1}{1 - s \sigma^2_j \lambda_k} \right)^{n_k N_x} \right]_{x = 1/(\sigma^2_j \lambda_k)} \\
    a_k &= \sigma^2_j \lambda_k/(1 + \sigma^2_j \lambda_k)
\end{align*}
\]

(22b)

\[
\begin{align*}
    P_{fa} &= \sum_{k=1}^{N} F_k Q_{X_{2a}}((2 \eta(1 + \sigma^2_j \lambda_k)/\sigma^2_j \lambda_k) \\
    F_k &= \prod_{i \neq k}^{N} \frac{1}{1 - (\lambda_i(1 + \sigma^2_j \lambda_k)/(1 + \sigma^2_j \lambda_k))}
\end{align*}
\]

(23b)

where \( B_k \)s are obtained using (22b). In the same manner (see (23b))

where again \( F_k \)s are calculated using (22b). It should be noted that (23a) and (23b) are the same as the results of [24, ch. 13]. Although (23) is derived for a special case, the interpretation of the detection performance is still difficult.

4.2.2 Single-input multiple-output (SIMO) radars (rank-1 signaling): In this case, the rank of \( P \) is one and \( \lambda_1 \) is the only non-zero eigenvalue of \( M^{-1/2} P P^H M^{-1/2} \) and hence \( N_x = 1, n_k = 1; \forall k \). Therefore one can simplify (22a) to (see (24))

Since \( Y(s) = (1 - 1 - 2 \sigma_j^2 \lambda_k)/2 \)^\(N_x\), calculation of \( C_j \) using (21a) results in \( C_j = 0 \) for \( j \neq N_x \) and \( C_{N_x} = 1 \). Note that the same condition also holds for \( C_j \). Also, it is inferred from (24) that in this circumstance, diversity is provided only by the receivers. For single-input single-output (SISO) case, (24) is reduced to the same results as in [24, ch. 13].

5 Bounds on detection performance

In this section we derive some lower and upper bounds on the probability of detection and the probability of false alarm using Chernoff bound (CB) to obtain better insight into detection performance in the general case. Generally, CB

\[
\begin{align*}
    P_d &= \sum_{k=1}^{N_x} C_k Q_{X_{2a}}((2 \eta/\sigma^2_j \lambda_k) = Q_{X_{2a}}((2 \eta/\lambda_1) \\
    P_{fa} &= \sum_{k=1}^{N_x} C_k Q_{X_{2a}}((2 \eta(1 + \sigma^2_j \lambda_k)/\sigma^2_j \lambda_k) = Q_{X_{2a}}((2 \eta(1 + \sigma^2_j \lambda_k)/\lambda_1))
\end{align*}
\]

(24)
can be expressed as follows [13, 26]

\[
\text{Prob}\{X \leq \gamma\} \leq \min_{\gamma \geq 0} \{\exp(\gamma \theta) E[\exp(-\gamma X)]\} \triangleq \min_{\gamma \geq 0} \{K(\gamma)\} \tag{25a}
\]

\[
\text{Prob}\{X \geq \gamma\} \leq \min_{\gamma \geq 0} \{\exp(-\gamma \theta) E[\exp(\gamma X)]\} \triangleq \min_{\gamma \geq 0} \{K(-\gamma)\} \tag{25b}
\]

It is worth noting that it has been shown that the stated optimisation problems in (25) are convex ones [26]. Therefore their unique global solutions could be available using efficient algorithms.

Since probability of missing a target, \( P_{\text{miss}} \), is

\[
P_{\text{miss}} = 1 - P_d = \text{Prob}\{T(x) \leq \eta | H_1\}
\]

\[
= \text{Prob}\left\{ \sum_{i=1}^{N_t} \sum_{k=1}^{N_c} (\sigma_{P,k,i}^2) \times \frac{1}{2} z_{k,i} \leq \eta \right\} \tag{26}
\]

using (25a), \( K(\gamma) \) can be expressed as follows

\[
K(\gamma) = \exp(\gamma \theta) E\left\{ \exp\left( -\gamma \left[ \sum_{i=1}^{N_t} \sum_{k=1}^{N_c} (\sigma_{P,k,i}^2) \times \frac{1}{2} z_{k,i} \right]\right) \right\}
\]

(27)

As \( z_{k,i}^2 \)s are \( \chi_2^2 \) RVs, (28) can be simplified using moment generation function (MGF) of these RVs [24] as

\[
\prod_{i=1}^{N_t} \prod_{k=1}^{N_c} E\left\{ \exp\left( -\gamma (\sigma_{P,k,i}^2) \times \frac{1}{2} z_{k,i} \right) \right\}
\]

\[
= \prod_{i=1}^{N_t} \prod_{k=1}^{N_c} \left( \frac{1}{1 + \gamma \sigma_{P,k,i}^2} \right) \forall \gamma \geq 0 \tag{29}
\]

Finally, \( P_{\text{miss}} \) may be bounded using (25), (27) and (29) as

\[
P_{\text{miss}} \leq \min_{\gamma \geq 0} \{K(\gamma)\}
\]

\[
= \min_{\gamma \geq 0} \left\{ \exp(\gamma \theta) \prod_{i=1}^{N_t} \prod_{k=1}^{N_c} \left( \frac{1}{1 + \gamma \sigma_{P,k,i}^2} \right) \right\} \tag{30a}
\]

\[
= \exp(\gamma \theta) \prod_{i=1}^{N_t} \prod_{k=1}^{N_c} \left( \frac{1}{1 + \gamma \sigma_{P,k,i}^2} \right) \tag{30b}
\]

where \( \gamma_{P,d}^{\text{opt}} \) is the solution of convex optimisation in (30a). Since \( P_{\text{miss}} = 1 - P_d \), (30) may be thought as a lower bound on \( P_d \). One may take into account the second factor in (30) and considers that employing more receivers and a higher target strength, \( \sigma_{P,k,i}^2 \), lead to the lower \( P_{\text{miss}} \) (note that always \( 1 + \gamma \sigma_{P,k,i}^2 \leq 1 \)). In addition, although the larger rank for \( A_s \) (the number of non-zero \( \lambda_{k,i} \) for fixed \( i \)) and larger values for \( \lambda_{k,i} \) do the same task, increasing both of them simultaneously is not possible in many real situations where total transmitted power is limited because total transmitted power can be expressed as trace(\( PP^H \)) leading to an upper bound on \( \sum_{\text{non-zero}\lambda_{k,i}} \lambda_{k,i} \). In essence, one may assume that a larger rank for \( A_s \) provides more diversity whereas a larger value for \( \lambda_{k,i} \) provides more signal-to-interference ratio for each diverse path. Investigation of this diversity-integration trade-off is not considered in the present paper and has been investigated thoroughly in [14].

It should be noted that an upper bound can also be derived for \( P_d \) by noting that

\[
P_d = \text{Prob}\{T(x) \geq \eta | H_1\}
\]

Therefore upper bound on \( P_d \) is available using (25b) and a procedure similar to (27)–(30)

\[
P_d \leq \min_{\gamma \geq 0} \{K(-\gamma)\}
\]

\[
= \left\{ \exp(-\gamma \theta) \prod_{i=1}^{N_t} \prod_{k=1}^{N_c} \left( \frac{1}{1 - \gamma \sigma_{P,k,i}^2} \right) \right\} \forall \gamma \in \Gamma^{(d)}
\]

(31a)

where \( \Gamma^{(d)} \) is the convergence set for (31a) and \( \gamma_{P,d}^{\text{opt}} \) is the optimum value of \( \gamma \) for upper bound on \( P_d \) obtained as the solution of minimisation of (31a) over \( \Gamma^{(d)} \). Since the involved RVs in (25b), that is, \( z_{k,i} \), are \( \chi_2^2 \) RVs, one can write \( \Gamma^{(d)} \) as (31b) considering convergence set of MGF of individual involved RVs

\[
\Gamma^{(d)} \triangleq \left[ 0, \min_{k=1,2,\ldots,N} \left( \frac{1}{1 + \gamma \sigma_{P,k,i}^2} \right) \right] \tag{31b}
\]

Finally, we have

\[
1 - \left\{ \exp(\gamma \theta) \prod_{i=1}^{N_t} \prod_{k=1}^{N_c} \left( \frac{1}{1 + \gamma \sigma_{P,k,i}^2} \right) \right\} \gamma_{P,d}^{\text{opt}} \leq P_d \leq \left\{ \exp(-\gamma \theta) \prod_{i=1}^{N_t} \prod_{k=1}^{N_c} \left( \frac{1}{1 - \gamma \sigma_{P,k,i}^2} \right) \right\} \gamma_{P,d}^{\text{opt}}
\]

(32)

It is worth noting that \( P_{fa} \) may be bounded using (25) under the \( H_0 \) in a method similar to the above that leads to

\[
1 - \left\{ \exp(\gamma \theta) \prod_{i=1}^{N_t} \prod_{k=1}^{N_c} \left( \frac{1}{1 + \gamma \sigma_{P,k,i}^2} \right) \right\} \gamma_{P,d}^{\text{opt}} \leq P_{fa} \leq \left\{ \exp(-\gamma \theta) \prod_{i=1}^{N_t} \prod_{k=1}^{N_c} \left( \frac{1}{1 - \gamma \sigma_{P,k,i}^2} \right) \right\} \gamma_{P,d}^{\text{opt}}
\]

(33a)

where \( \gamma_{P,d}^{\text{opt}} \) and \( \gamma_{P,d}^{\text{opt}} \) are optimum values of \( \gamma \) for lower
and upper bounds in (33a), respectively. Owing to the fact that \( z_{\Delta}^T \) are \( \chi^2 \) RVs, for the lower CB in (33a), \( \gamma \) can be any non-negative value while for the upper CB in (33a) it should belong to the below set

\[
\Gamma^{(a)} \triangleq \left\{ 0, \min_{k=1,2,\ldots,N_r} \left( 1 + \sigma_T^2 \Lambda_{k,1}^f \right)/(\sigma_T^2 \Lambda_{k,1}) \right\}
\]

(33b)

where, similar to (31b), \( \Gamma^{(a)} \) is obtained using MGF of involved RVs.

6 Numerical results

In this section, several simulation results are drawn in order to investigate and validate derived results in the previous sections. In the first part, a comparison between the theoretical and simulated performance analysis is made. The second part contains an investigation of the effect of various parameters on the detection performance. In all the simulations, we consider a situation in which total transmitted power is limited and SNR is defined as 10 log(\( \sigma_T^2 \)trace(PP\( ^T \)))/trace(M).

Coloured interference is modelled as a CSGRP with known covariance matrices, \( M_i, i = 1, 2, \ldots, N_r \), as stated in Section 2. In addition, it is assumed that \( M_i \) has been shaped exponentially with a lag-parameter equal to \( \rho \), that is, \( M_i[\mu_\rho] = \rho^{m-n} \).

Furthermore, without loss of generality, it is assumed that \( \sigma_T^2 = 1 \) and \( \text{SNR} = 3 \text{ dB} \). Also transmitted waveforms are assumed to be random \( (P \text{ is assumed to be a random matrix}) \) unless it is explicitly mentioned.

6.1 Comparison of the theoretical and simulated detection performance

The theoretical and simulated detection performances are compared in this part. It is worth noting that all simulated probabilities are estimated using Monte-Carlo simulations as proposed in [24]. Note that in some parts of the following figures, \( P_d \) and \( P_{fa} \) are drawn against detection threshold, that is, \( \eta \) in (10), just for confirmation of the derived formulas for them.

Fig. 1 shows the theoretical and simulated detection performance in the most general case stated in Section 4.1. Fig. 1a shows the theoretical detection probability of (15) and also the simulated one against the detection threshold. Also, this figure contains lower and upper CBs stated in (32). In this case, an MIMO scenario with \( N = 2, N_r = 2, N_c = 3, p_1 = 0.8, p_2 = 0.7 \) and \( p_3 = 0.7 \) has been simulated. In this condition, the sequence of \( \{\Lambda_{k,1}\} \) is \( [0.2459, 0.2459, 0.3669, 6.6925, 6.70485, 7.0485] \).

Therefore the sequence of \( \{\lambda^{(d)}_{p,w}\} \) is defined as \( [0.2459, 0.3669, 6.6925, 7.0485] \) with multiplicity, that is, \( m_{p,w} \) is of \( \{2, 1, 1, 2\} \).

It is evident from Fig. 1a that discrimination of the theoretical and simulated \( P_d \) is difficult. In addition, lower CB tracks \( P_d \) for lower threshold in a better way while the condition for upper CB is the opposite (similar behaviour of CBs against SNR can be seen in [13], where CBs have been shown for GLRT detector). In Fig. 1b, a zoomed version of the theoretical and simulated \( P_d \) is provided to emphasise the existence of very minor differences. The investigations of these subjects for the false alarm probability are shown in Figs. 1c and d, respectively. Precisely, the theoretical false alarm probability, \( P_{fa} \), derived in (16), simulated one and CBs stated in (33), are shown in Fig. 1c. Minor dissimilarities between the theoretical and simulated \( P_{fa} \) are highlighted in Fig. 1d. Finally, Fig. 1e contains the theoretical and simulated receiver operating characteristics (ROC). It can be inferred from this figure that simulated ROC is less compatible with the theoretical one in the last simulated decay, a well-known behaviour of simulated ROCs. It should be noted that although exact closed-form expressions for \( P_d \) and \( P_{fa} \) are derived in this paper for statistical MIMO radar in (15) and (16), derivation of a closed-form expression for ROC is not possible due to the complicated form of (15) and (16). Consequently, the theoretical ROC shown in Fig. 1e is obtained by numerical elimination of threshold, \( \eta \), from (15) and (16).

The situation in which covariance matrices of the interference for all receivers are identical is investigated in Fig. 2. In this case, an MIMO scenario with \( N = 2, N_r = 2, N_c = 2 \) has been simulated where \( \rho = 0.8 \) leading to \( \lambda_t = [0.1554, 4.8377] \) and \( n_k = \{1, 1\} \). The theoretical \( P_d \) derived by (22), the simulated one and CBs are drawn in Fig. 2a while the same graphs for \( P_{fa} \) are shown in Fig. 2c. Minor differences between the theoretical and simulated probabilities are presented in Figs. 2b and d for \( P_d \) and \( P_{fa} \), respectively. Fig. 2e shows the theoretical and simulated ROCs. Inferred consequences are similar to those of the previous figure; especially, consistency of the theoretical derivations with the simulation results can be seen again in this figure.

To verify the trustiness of derived formula for MISO case in (23) and also the SIMO case in (24), we rely on the comparison of the theoretical and simulated ROCs. Figs. 3a and b show the theoretical and simulated ROCs for MISO and SIMO systems where \( N = 2, N_r = 2, N_c = 1 \) and \( N = 2, N_r = 1, N_c = 2 \), respectively. Again, it can be inferred that the theoretical and simulated results are highly consistent especially in the first decays of simulated ROCs which are more reliable.

6.2 Effect of various parameters on the detection performance

In this part, the effect of various parameters on the detection performance is investigated. Considering the test statistic in (10), it can be inferred that the detection performance will be affected by the transmission waveforms via \( P \), the interference covariances, the target strength and the number of receivers. As the effect of the target strength, \( \sigma_T^2 \), is clear, only the effect of others will be considered in the following.

Fig. 4 shows the effect of the number of receivers on the detection performance for an MIMO system with \( N = 2 \) and \( N_r = 2 \), which employs a fixed random \( P \) for \( N_r = 1, 2, 3 \). Also, it is assumed that the interference covariances are identical with \( \rho = 0.8 \).

It is evident from Fig. 4 that increasing the number of receivers will improve detection performance. This can be explained by considering (19) where the degrees of freedom of the summing chi-square RV are equal to \( 2N_c \) (identical covariances). For the general case, increasing the \( N_r \) increases the summing terms in (13) leading to a better performance. This may be explained roughly using CBs for \( P_d \) in (30) where larger \( N_r \) results in higher lower CB for the detection probability.

The effect of the interference correlation is considered in Figs. 5a and b for an MIMO system with \( N = 2, N_r = 2 \).
Fig. 1  Theoretical and simulated detection performance in general case

a Theoretical $P_d$, simulated $P_d$ and CBs for general case

b Highlighting the differences between the theoretical and simulated $P_d$ for general case

c Theoretical $P_{fa}$, simulated $P_{fa}$ and CBs for general case

d Highlighting the differences between the theoretical and simulated $P_{fa}$ for general case

e Theoretical and simulated ROCs for general case
Fig. 2  Theoretical and simulated detection performance in identical covariance matrices for interference

a  Theoretical $P_d$, simulated $P_d$ and CBs for identical interference
b  Highlighting the differences between the theoretical and simulated $P_d$ for identical interference
c  Theoretical $P_{fa}$, simulated $P_{fa}$ and CBs for identical interference
d  Highlighting the differences between the theoretical and simulated $P_{fa}$ for identical interference
e  Theoretical and simulated ROCs for identical interference
and $N_r = 2$ for two random $P_s$, respectively. Fig. 5a shows ROC of the detector for both white and coloured interference with $\rho = 0.8$ for $P_1$. As it is obvious from this figure, detector performance is better in white interference than that of coloured interference.

One can see the reverse state in Fig. 5b, where the detector performance is better in coloured interference than that of white interference for $P_2$. It can be concluded that the effect of the interference correlation on the detection performance is dependent on the transmitted signals and therefore, design of transmission signals plays an important role in the detector performance especially in coloured interference. This issue has been discussed thoroughly in [13, 14]. An important result is that we can improve the detection performance of the detector in coloured interference (with known covariance) by utilising proper transmission signal.

The effect of the number of transmitters is analysed in Fig. 6 for an MIMO system with $N = 2$, $N_r = 2$ and $\rho = 0.8$. It is assumed that the system is uncoded, meaning that its policy is to transmit pulses with equal amplitude

**Fig. 3** Theoretical and simulated ROCs for MISO and SIMO systems

- a Comparison of the theoretical and simulated ROCs for MISO systems
- b Comparison of the theoretical and simulated ROCs for SIMO systems

**Fig. 4** Investigation of the effect of the number of receivers

**Fig. 5** Effect of the interference correlation

- a Effect of the interference correlation for the first transmitted signal
- b Effect of the interference correlation for the second transmitted signal
results in that \( P \) is a scaled version of all-one matrix. It is seen from Fig. 6 that increasing the \( N_t \) does not improve the detection performance and systems with one and two transmitters are identical; in essence, the system with two transmitters is unable to utilise available resources. The reason for the resulting phenomenon originates from the fact that for both \( N_t = 1, 2 \) the rank of corresponding \( P \) is equal to one resulting in rank-1 signalling [13, 14]. Note that the total transmitted power in these two cases is fixed.

It should be noted that although adding more transmitters to the system will increase upper limit of summations in (10), (13) and (19), the actual number of summing terms is dependent on the number of non-zero eigenvalues, that is, \( \lambda_{k,i} \), determined by the rank of \( 1/(\sigma_i^2)A = M_k^{1/2}PP^H M_k^{1/2} \) for \( i = 1, 2, \ldots, N_r \) and affected by transmission signals, \( P \). Therefore increasing the number of transmitters does not necessarily increase the number of summing terms in (13) as opposed to increasing the number of receivers. Consequently, as choosing transmission signals affects both the number of non-zero eigenvalues and their values, it plays an important role in detection performance. In other words, it is not useful to increase the number of transmitters in blind manner because the total transmitted power is limited and this limitation imposes an upper bound on the sum of the \( \lambda_{k,i} \). In essence, available power should be distributed properly among the transmitters. This issue in the general condition has been investigated comprehensively in [13, 14].

7 Conclusion

Theoretical performance analysis of optimum detector in statistical MIMO radar was carried out in this paper. After a brief introduction of the problem formulation, optimum detector for coloured Gaussian interference, multi-pulse detection and arbitrary transmission was developed. Exact closed-form expressions for the probability of detection and also false alarm in the stated situation were obtained. Although the presented expressions were derived for the mentioned modelling, proposed lemma can be used to calculate detection performance of the optimum detector for any Gaussian problem, for example, collocated MIMO modelling in [21]. Owing to the fact that the derived expressions are complicated and thus, not easy to interpret, lower and upper CBs were also proposed to help better understanding of detector performance. Monte-Carlo simulations were carried out in various situations, and the agreement between results showed the validity of theoretical analysis. Investigation of the effect of various parameters on detection performance reveals that increasing the number of receivers always improves detection performance, whereas the effect of the number of transmitters on that is highly dependent on the transmission signals. Furthermore, it was observed that transmission signals play an important role in the effect of interference correlation on the detector performance. Consequently, although the design of optimum transmission signals was out of the scope of this paper, it was pointed out that it has a significant effect on the detection performance and can be subject of the future work.

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9 References

1 Li, J., Stoica, P.: ‘MIMO radar signal processing’ (Wiley, 2009)
10 Appendix 1: proof of Lemma 1

Since \( h_{1,s} \) are \( \chi^2_2 \) i.i.d RVs, MGF of \( g_{l,1}h_{i,1} \), becomes \( (1/(1-2sg_{l,1})) \) [24]. Owing to the independency of \( h_{1,s} \), for MGF of \( z = \sum_{l=1}^{L} \sum_{l=1}^{M} g_{l,i}h_{i,t} \), one can write

\[
\prod_{j=1}^{M} \prod_{t=1}^{2} \frac{1}{1-2sg_{l,j}} = (1-2sg_{l,1})(1-2sg_{l,2}) \ldots (1-2sg_{l,M})
\]

(34)

Now, define a new sequence with distinct elements denoted by \( \{b_{n,k}\}_{k=1}^{L} \ldots \} \) \( \{g_{l,j}\}_{l=1}^{L} \). In other words, the new sequence, that is, \( \{m_{n,k}\}_{k=1}^{L} \ldots \}_{l=1}^{M} \), shows the multiplicity of \( \{b_{n,k}\}_{k=1}^{L} \ldots \} \) \( \{g_{l,j}\}_{l=1}^{L} \).

Therefore (34) can be rewritten as

\[
\prod_{n=1}^{N} \prod_{k=1}^{M} \left( \frac{1}{1-2sb_{n,k}} \right)^{m_{n,k}} \frac{1}{(1-2sb_{1,1})(1-2sb_{1,2}) \ldots (1-2sb_{n,n})}
\]

\( \triangleq Y(s) \)

Using the partial fraction expansion, we have

\[
\prod_{n=1}^{N} \prod_{k=1}^{M} \left( \frac{1}{1-2sb_{n,k}} \right)^{m_{n,k}} \frac{1}{(1-2sb_{1,1})} \right) \frac{D_{1,1,i}}{m_{n,k}} + \ldots + \sum_{j=1}^{m_{n,k}} \frac{D_{n,k,1,i}}{(1-2sb_{n,k})^{j}}
\]

(35)

Consequently, (35) can be recasted as (see (36))

Therefore by taking the inverse Laplace transform of (36), the PDF of \( z \) is available as the following

\[
f_{z}(z) = \text{Laplace}^{-1} \left\{ \sum_{n=1}^{N} \sum_{k=1}^{M} \sum_{j=1}^{m_{n,k}} \frac{D_{n,k,1,i}}{(1-2sb_{n,k})^{j}} \right\} \bigg|_{z \rightarrow z}
\]

(36)

11 Appendix 2: proof of Corollary 1

In Lemma 1, assume that \( g_{l,i} = g_{l,j} \) \( \forall l = 1, 2, \ldots, L \). Thus, \( z = \sum_{l=1}^{L} \sum_{t=1}^{M} g_{l,i}h_{i,t} = \sum_{l=1}^{L} \sum_{t=1}^{M} h_{i,t} \). Since \( h_{l,s} \) are independent RVs for different \( l \), \( \sum_{t=1}^{M} h_{i,t} \) is a \( \chi^2_M \) RV and we have

\[
\left\{ a_{i} \triangleq g_{l,i} \right\}
\]

(39)

Therefore using (39), (37) and (38), for \( c = \sum_{i=1}^{M} a_{i}y_{i} \), one can write

\[
\frac{f_{z}(z)}{z} = \sum_{k=1}^{M} A_{k} \left( \frac{1}{(Lu_{k} - j)} \right)^{M} \left( (1-2sb_{n,k})^{m_{n,k}} \right) \frac{D_{n,k,1,i}}{m_{n,k}}
\]

(37)

\[
\frac{Q_{C}(z)}{z} = \text{Prob}\{ C \geq z \} = \int_{z}^{\infty} f_{z}(x)dx = \sum_{n=1}^{N} \sum_{k=1}^{M} D_{n,k,1,i} Q_{z/n}(z/b_{n,k})
\]

(38)