Sampling theorem

- Notes:
  - Discrete-time frequency
    - Normalized
    - Radian
    - Maximum frequency
  - Continuous-time frequency
    - Radian/sec or Hz

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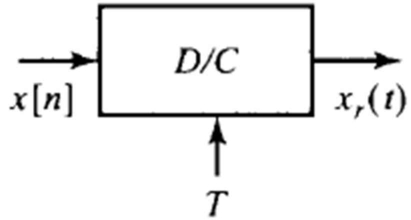
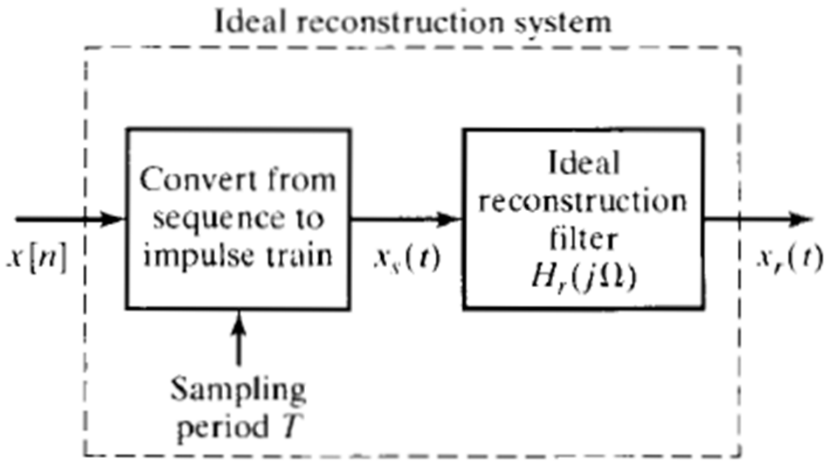
Sampling theorem

- Remark on D/C:
  - Remember that



$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT).$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$



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Sampling theorem

- Therefore, CTFT of both of

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT).$$

  – Gives

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] H_r(j\Omega) e^{-j\Omega T n}$$

  – Now, observe that

$$X_r(j\Omega) = H_r(j\Omega) X(e^{j\Omega T})$$

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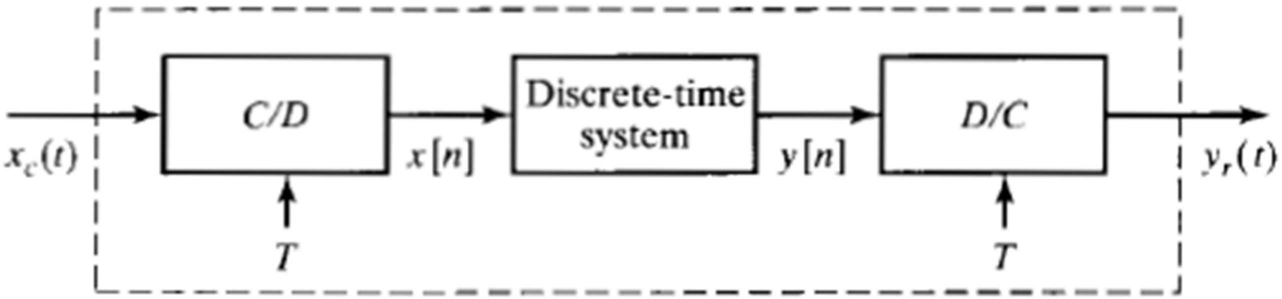
Sampling theorem

- Discrete-time processing of the continuous-time signals



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Sampling theorem

- We know that

$$x[n] = x_c(nT).$$

  – and the relationship between FT

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

  – and from reconstruction (interpolation)

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T},$$

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Sampling theorem

- Hence, for the DT processing of CT signals we have (see ideal D/C):

$$Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T})$$
$$= \begin{cases} TY(e^{j\Omega T}), & |\Omega| < \pi/T, \\ 0, & \text{otherwise.} \end{cases}$$

- Special case: the identity system $y[n] = x[n] = x_c(nT)$
  - then $y_r(t) = x_c(t)$

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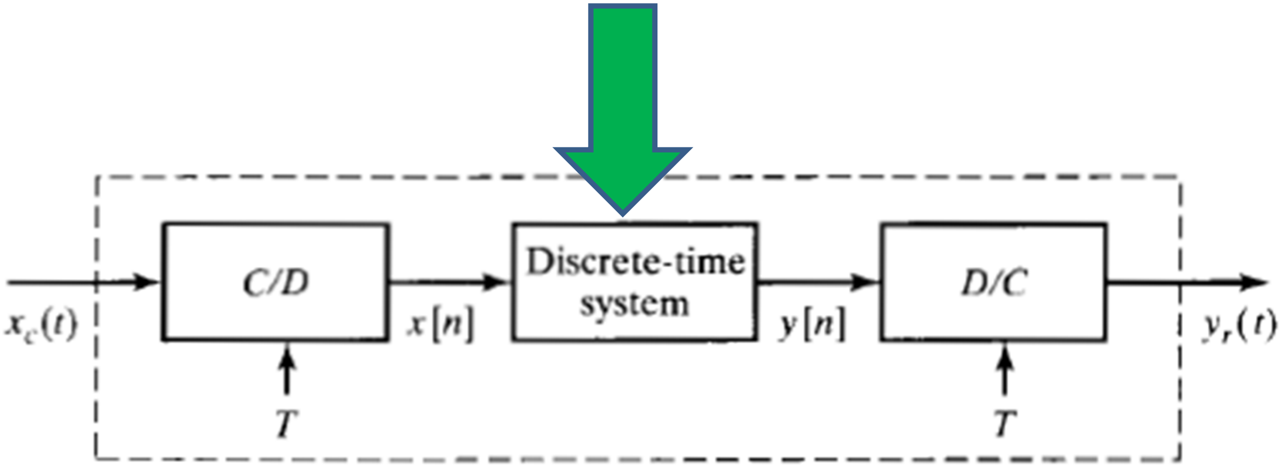
Sampling theorem

- We consider an DT LTI system:



$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T})$$

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Sampling theorem

- We had

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

  - Which results in ($\omega = \Omega T$)

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

    - Double check sampling frequency….

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Sampling theorem

- Reasonably

$$X_c(j\Omega) = 0 \text{ for } |\Omega| \geq \pi/T.$$

- Therefore

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega), & |\Omega| < \pi/T, \\ 0, & |\Omega| \geq \pi/T. \end{cases}$$

- Finally

$$Y_r(j\Omega) = H_{\text{eff}}(j\Omega)X_c(j\Omega),$$

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T, \\ 0, & |\Omega| \geq \pi/T. \end{cases}$$

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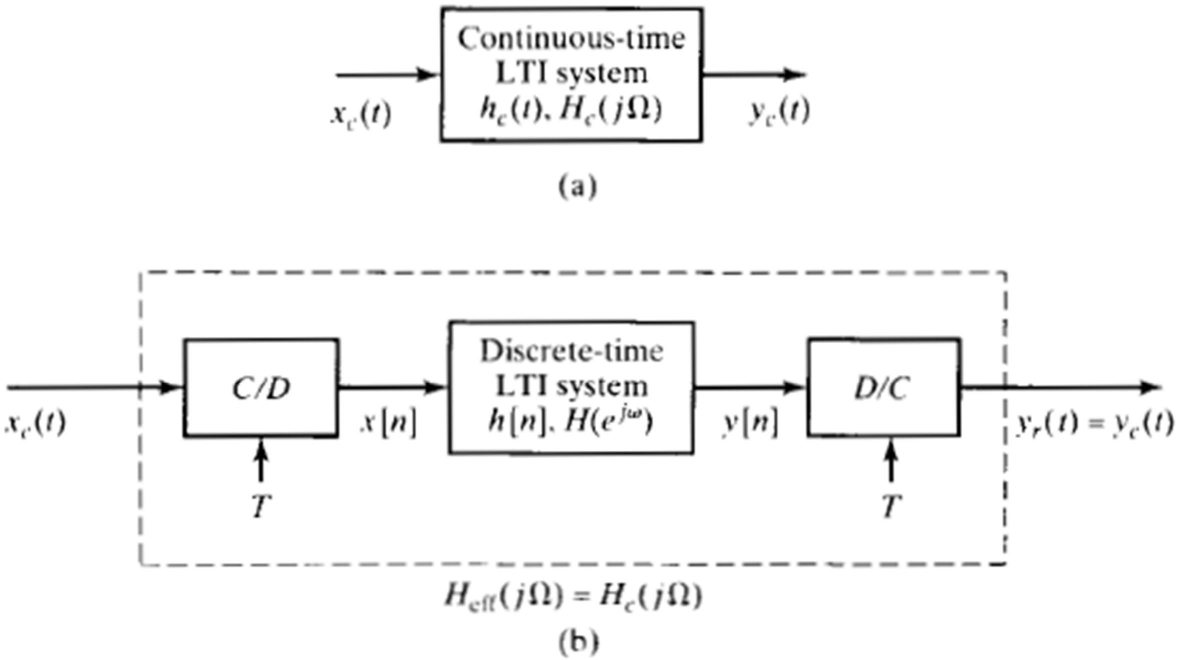
Sampling theorem

- Detailed diagram



(a) Continuous-time LTI system. (b) Equivalent system for bandlimited inputs.

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Sampling theorem

- Note: important assumptions
  - An LTI DT system
  - Band-limited input signal $x_c(t)$
  - Sampling rate above the Nyquest rate
- Then the diagram can be modeled via a CT LTI system

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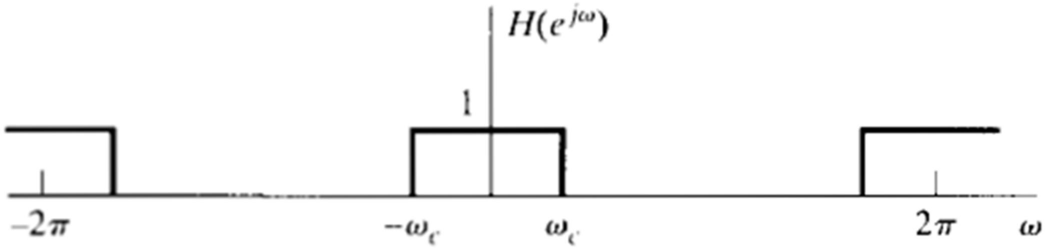
Sampling theorem

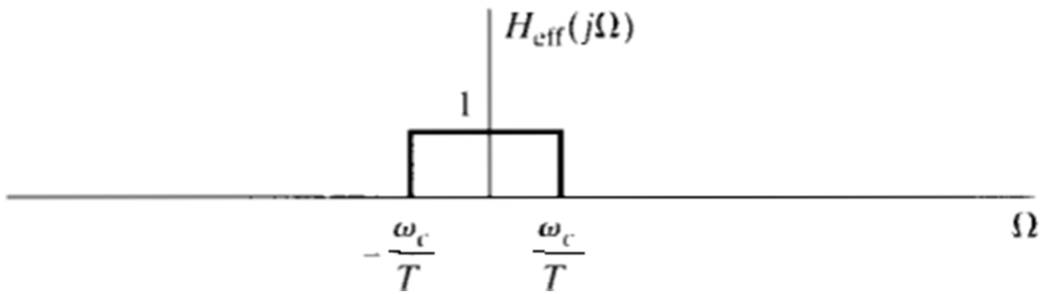
- Example: ideal LPF

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi. \end{cases}$$



- Then,

$$H_{\text{eff}}(j\Omega) = \begin{cases} 1, & |\Omega T| < \omega_c \text{ or } |\Omega| < \omega_c/T, \\ 0, & |\Omega T| > \omega_c \text{ or } |\Omega| > \omega_c/T. \end{cases}$$



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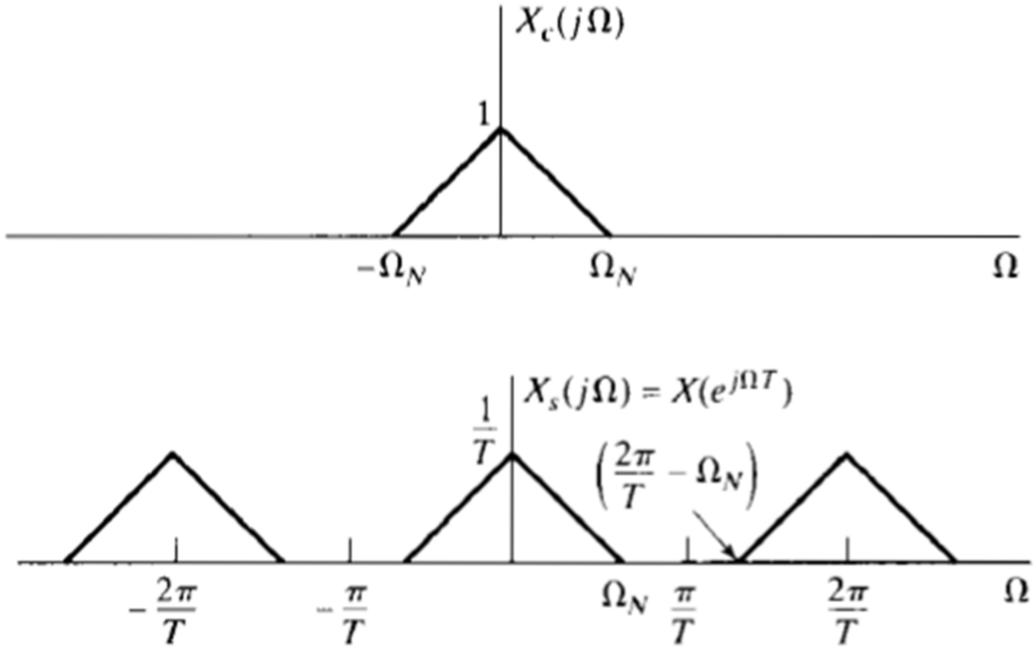
Sampling theorem

- LPF of CT signals via DT ideal LPF



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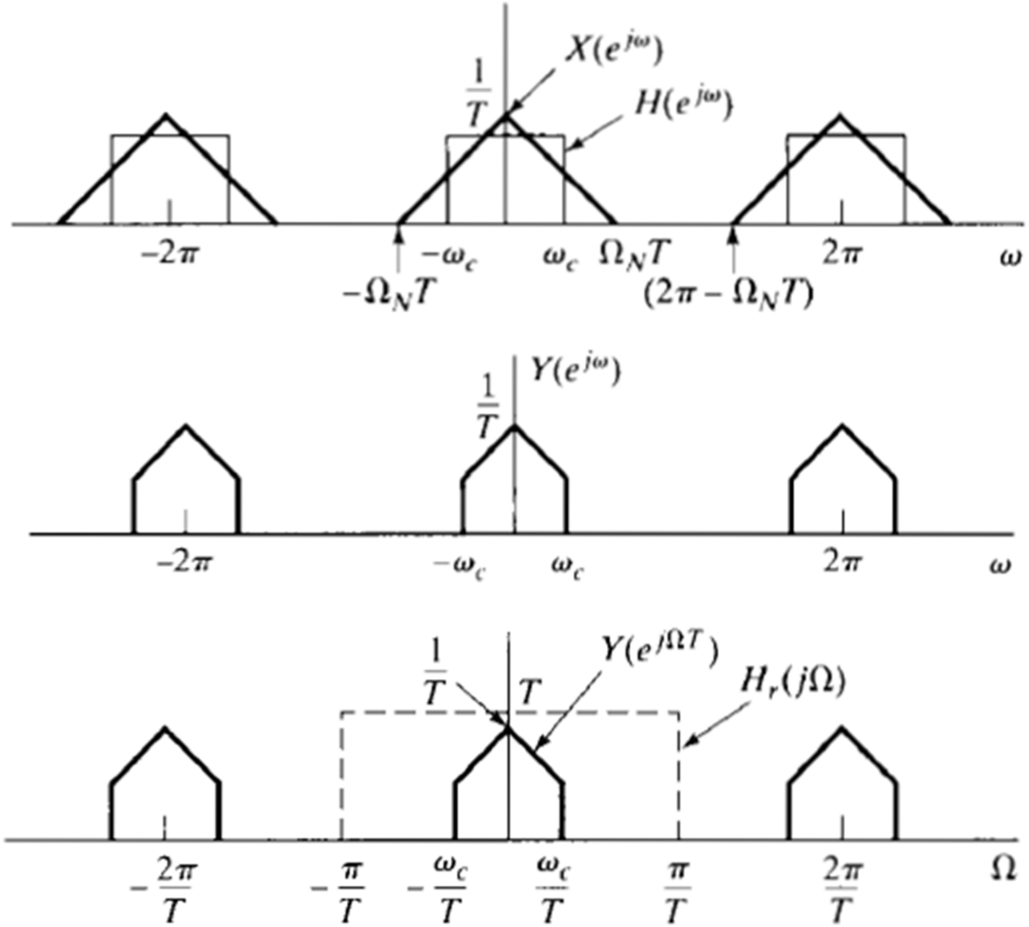
Sampling theorem

- And,

$$(2\pi - \Omega_N T) > \Omega_N T.$$



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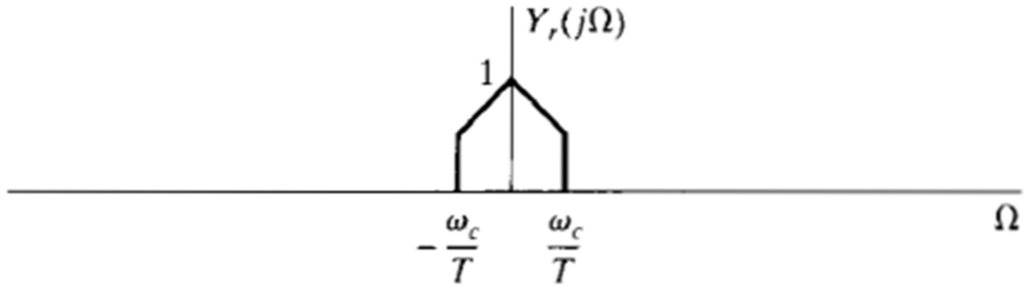
Sampling theorem

- Finally, the output is given by



- Observation: cut-off frequency of the CT LPF
  - Sampling frequency
  - DT cut-off frequency

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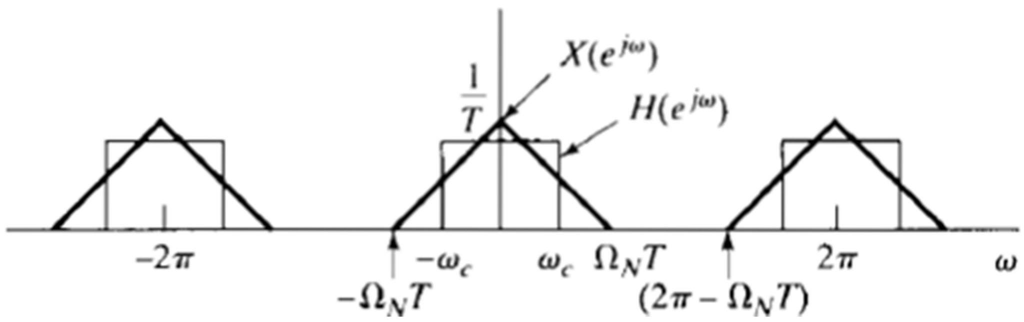
Sampling theorem

- Example: select sampling period such that

$$\Omega_N T < \omega_c$$

  – Herein, $y_r(t) = x_c(t)$



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Sampling theorem

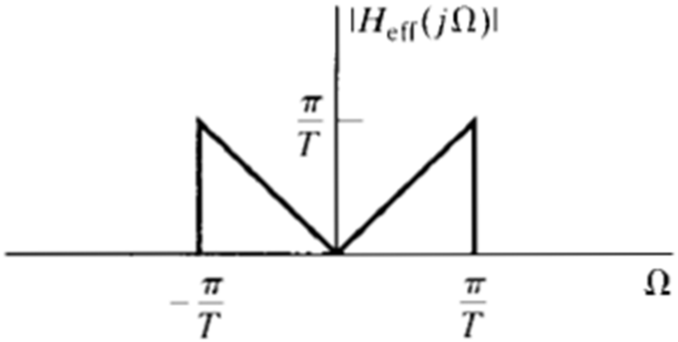
- Example: differentiator
  - We know that

$$y_c(t) = \frac{d}{dt}[x_c(t)].$$

$$H_c(j\Omega) = j\Omega$$

  - Also,

$$H_{\mathrm{eff}}(j\Omega) = \begin{cases} j\Omega, & |\Omega| < \pi/T, \\ 0, & |\Omega| \geq \pi/T, \end{cases}$$



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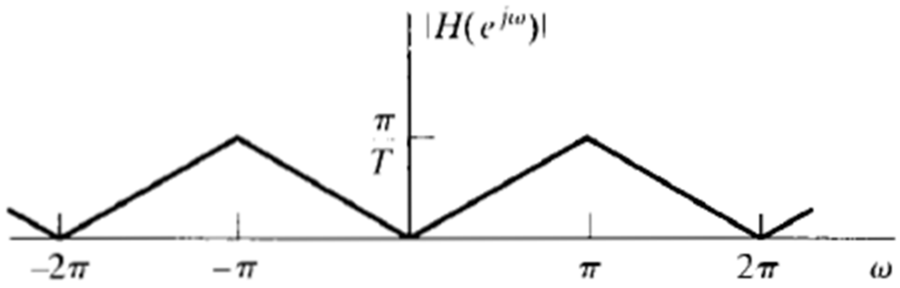
Sampling theorem

- The corresponding DT system

$$H(e^{j\omega}) = \frac{j\omega}{T}, \qquad |\omega| < \pi,$$



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Sampling theorem

– With the impulse response

$$h[n] = \frac{\pi n \cos \pi n - \sin \pi n}{\pi n^2 T}, \qquad -\infty < n < \infty,$$

– Or

$$h[n] = \begin{cases} 0, & n = 0, \\ \dfrac{\cos \pi n}{nT}, & n \neq 0. \end{cases}$$

– And the implementation of CT differentiator is done in CT domain!

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Sampling theorem

- Example: differentiation of sin

$$x_c(t) = \cos(\Omega_0 t)$$

$$\Omega_0 < \pi/T.$$

$$x[n] = \cos(\omega_0 n)$$

$$\omega_0 = \Omega_0 T < \pi$$

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty}[\pi\delta(\Omega - \Omega_0 - k\Omega_s) + \pi\delta(\Omega + \Omega_0 - k\Omega_s)].$$

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Sampling theorem

- And for the baseband,

$$X(e^{j\Omega T}) = \frac{\pi}{T}\delta(\Omega - \Omega_0) + \frac{\pi}{T}\delta(\Omega + \Omega_0) \qquad \text{for } |\Omega| \leq \pi/T.$$

  – Using $\Omega = \omega/T$

$$X(e^{j\omega}) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0), \qquad |\omega| \leq \pi.$$

    - Prove that $\delta(\omega/T) = T\delta(\omega).$

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Sampling theorem

- Always remember that
- $X(e^{j\omega})$ is repeated with period 2π
- $X(e^{j\Omega T})$ is repeated with period $2\pi/T$.
- Straightforwardly:

$$
\begin{aligned}
Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\
&= \frac{j\omega}{T}[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)] \\
&= \frac{j\omega_0\pi}{T}\delta(\omega-\omega_0)-\frac{j\omega_0\pi}{T}\delta(\omega+\omega_0), \qquad |\omega|\le\pi.
\end{aligned}
$$

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Sampling theorem

- And,

$$\begin{aligned} Y_r(j\Omega) &= H_r(j\Omega)Y(e^{j\Omega T}) = TY(e^{j\Omega T}) \\ &= T\left[\frac{j\omega_0\pi}{T}\delta(\Omega T - \Omega_0 T) - \frac{j\omega_0\pi}{T}\delta(\Omega T + \Omega_0 T)\right] \\ &= T\left[\frac{j\omega_0\pi}{T}\frac{1}{T}\delta(\Omega - \Omega_0) - \frac{j\omega_0\pi}{T}\frac{1}{T}\delta(\Omega + \Omega_0)\right] \\ &= j\Omega_0\pi\,\delta(\Omega - \Omega_0) - j\Omega_0\pi\,\delta(\Omega + \Omega_0). \end{aligned}$$

  – Which means

$$y_r(t) = j\Omega_0\frac{1}{2}e^{j\Omega_0 t} - j\Omega_0\frac{1}{2}e^{-j\Omega_0 t} = -\Omega_0\sin(\Omega_0 t) = \frac{d}{dt}\left[x_c(t)\right]$$

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Sampling theorem

- Impulse invariance
- Sampling of the impulse response of LTI CT systems

$$h[n] = T h_c(nT),$$

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T}\right), \qquad |\omega| \le \pi.$$

- Example: LPF

$$\Omega_c = \omega_c/T < \pi/T$$

$$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c, \\ 0, & |\Omega| \le \Omega_c. \end{cases}$$

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Sampling theorem

- The impulse response of the CT LPF:

$$h_c(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

- The impulse response of DT system

$$h[n] = Th_c(nT) = T\frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n}$$

- And,

$$\omega_c = \Omega_c T$$

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

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Sampling theorem

- Anti-aliasing filter

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