Digital Signal Processing (DSP)

Fall 2014

Isfahan University of Technology
Mohammad Mahdi Naghsh
mm_naghsh@cc.iut.ac.ir

DIGITAL SIGNAL PROCESSING (DSP)

Lecture 3
Discrete-time Fourier Transform (DTFT)

Mohammad Mahdi Naghsh ECE. Dept.,
Isfahan University of Technology

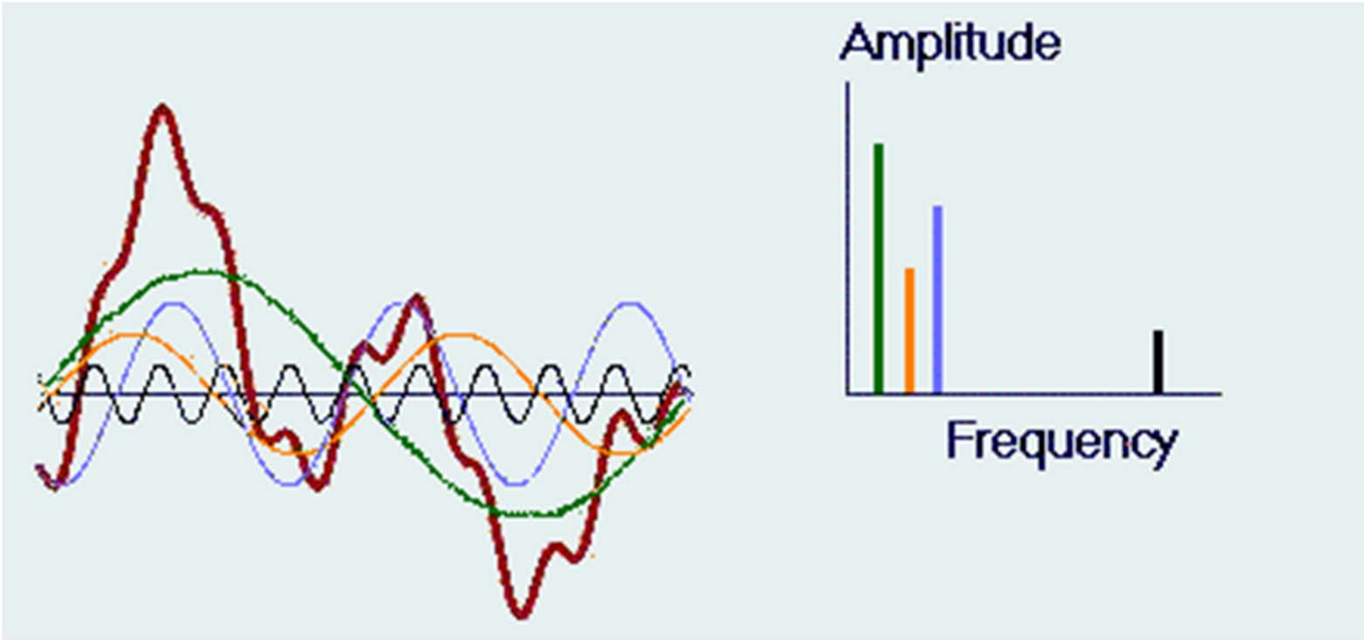
DTFT

- Frequency analysis vs. time analysis
  - An alternative tool
  - Equal information content
  - Fourier series, CT Fourier transform
    - Joseph Fourier: a French teacher
  - Expansion using the sin(.) & cos(.)
  - Single tone: frequency content

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

- Example: sum of sinusoidal signals



http://www.qsl.net/on7yd/136narro.htm

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

- Interpretation of the sum: difficult in time domain
  - An easy task in frequency domain
  - Delta functions corresponding to each sin(.)
  - Determine frequency of sin(.)
  - Determine amplitude of sin(.)
  - Bolding some features!!
- A useful tool for LTI system analysis

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

- Definition
  - A transform that maps the DT signal x[n] into the following continuous function (spectrum):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  - We use the notation $X(e^{j\omega})$ to emphasis on periodicity w.r.t. ω
  - Generally complex-valued $X(e^{j\omega})$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

- Note

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

  – Or

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}$$

  – (X), dB: 20log(X)
  – and

$$X(e^{j\omega+2\pi}) = X(e^{j\omega})$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

- Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega,$$

  – Observe the periodicity
  – The frequency content interpretation
  – DTFT pair: analysis & synthesis

$$\{x[n]\} \leftrightarrow X(e^{j\omega})$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

- Example: $\{x[n]\} = \{a^n u[n]\}$
  - by definition,

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{\infty} e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (ae^{-j\omega})^n \\ &= \frac{1}{1 - ae^{-j\omega}} \end{aligned}$$

  - exists when $|ae^{-j\omega n}| < 1$ i.e. $|a| < 1$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

- Convergence: an infinite sum!!
- Define the partial sum

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}$$

  - Case 1(absolutely summable signals) the partial sum converges to the spectrum
  - Case 2(energy signals): DTFT exists (but with a weaker statement)

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

– Example (case 1): absolutely summability of the previous example

$$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} < +\infty$$

- if $|a| < 1$

– Example (case 2-energy signals): Gibbs phenomenon

$$h_{\text{lp}}[n] = \frac{\sin \omega_c n}{\pi n}$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

– The partial sum convergence

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

– The limit of the partial sum

$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

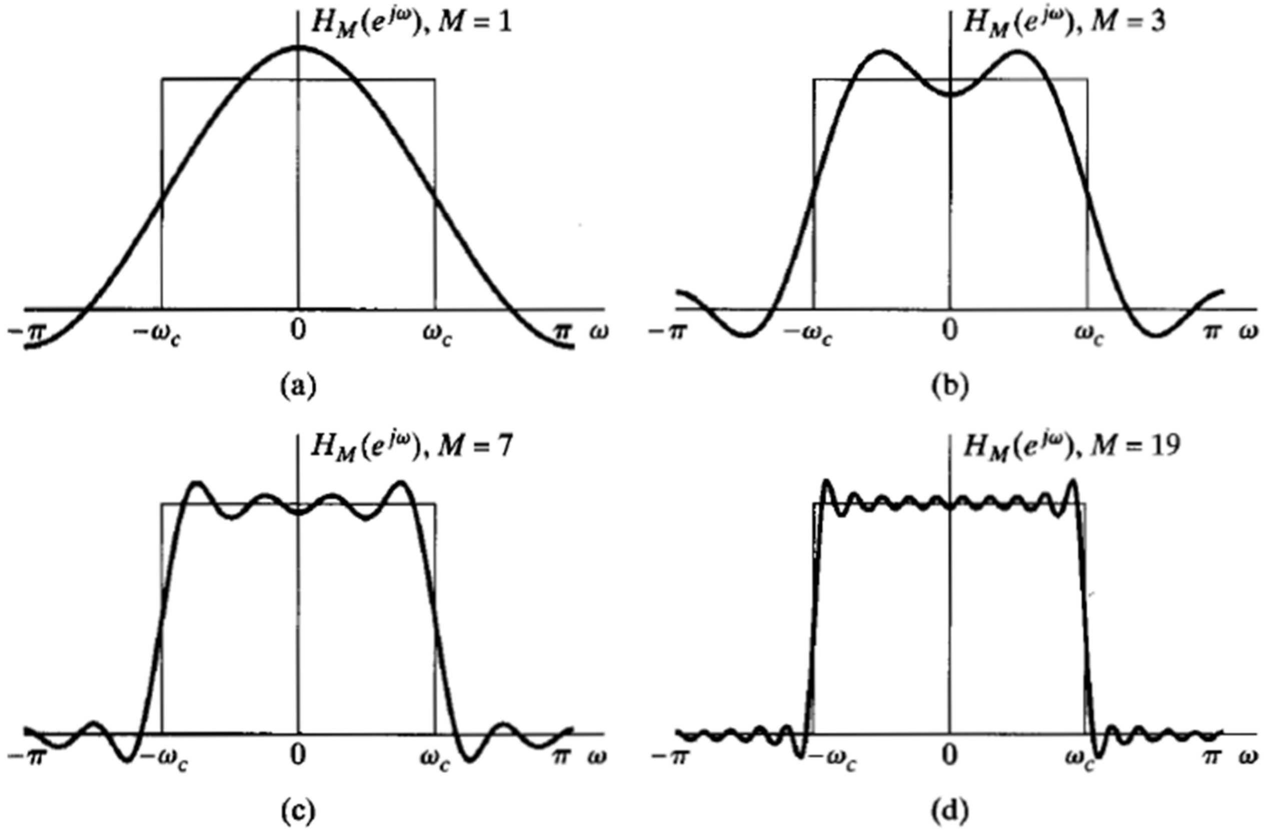
– Indeed,

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} |H_{\mathrm{lp}}(e^{j\omega}) - H_M(e^{j\omega})|^2 d\omega = 0.$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

– Illustration: Gibbs



Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

– Case 3(power signals): may converge

- The results contain delta functions

(a) $\int_{-\infty}^{\infty} \delta(\omega) d\omega = 1$

(b) $\int_{-\infty}^{\infty} X(e^{j\omega})\delta(\omega-\omega_0) d\omega = X(e^{j\omega_0})$ if $X(e^{j\omega})$ is continuous at $\omega = \omega_0$;

(c) $X(e^{j\omega})\delta(\omega) = X(e^{j0})\delta(\omega)$ if $X(e^{j\omega})$ is continuous at $\omega = 0$

- Note also that $\delta(\omega) = 0$ for $\omega \neq 0$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

- Example (DTFT of a constant): $x[n] = 1 \text{ for all } n$
  - Claim

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$$

  - Proof via the synthesis equation

$$\begin{aligned} x[n] &= \int_{-\pi}^{\pi} \sum_{h=-\infty}^{\infty} \delta(\omega + 2\pi h) e^{j\omega n} d\omega \\ &= \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega \;\; = 1 \end{aligned}$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

- Example (DTFT of CES): $x[n] = e^{j\omega_0 n}$
  - Claim

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 + 2\pi k), \quad -\pi < \omega_0 \leq \pi$$

  - Similar proof

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{h=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi h) e^{j\omega n} d\omega \\ &= \int_{-\pi}^{\pi} \delta(\omega - \omega_0) e^{j\omega n} d\omega \end{aligned}$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

- Properties of DTFT
  - Periodicity

As noted earlier that the DTFT $X(e^{j\omega})$ is a periodic function of ω with period 2π. This property is different from the continuous time Fourier transform of a signal.

  - Linearity

$$\begin{aligned}&\text{If } \{x[n]\} \leftrightarrow X(e^{j\omega})\\ &\text{and } \{y[n]\} \leftrightarrow Y(e^{j\omega})\\ &\text{then } a\{x[n]\} + b\{y[n]\} \leftrightarrow aX(e^{j\omega}) + bY(e^{j\omega})\end{aligned}$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

– Conjugation

$$\{x^*[n]\} \leftrightarrow X^*(e^{-j\omega})$$

- A simple proof

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} &= \sum_{n=-\infty}^{\infty} [x[n]e^{j\omega n}]^* \\ &= \left[\sum_{n=-\infty}^{\infty} x[n]e^{-j(-\omega)n}\right]^* \\ &= X^*(e^{-j\omega})\end{aligned}$$

– Time reversal

$$\{x[-n]\} \leftrightarrow X(e^{-j\omega})$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

- Time shifting

$$\{x[n-n_0]\} \leftrightarrow e^{-j\omega n_0} X(e^{j\omega})$$

- Modulation

$$\{e^{j\omega_0 n} x[n]\} \leftrightarrow X(e^{j(\omega-\omega_0)})$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

DTFT

- Symmetry property (Hermitian symmetry)
  - Let x[n] is real, then

$$x[n] = x^*[n]$$

    - which leads to the following symmetry

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

    - Proof : straightforward
    - Useful/important results

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology