Filter Design

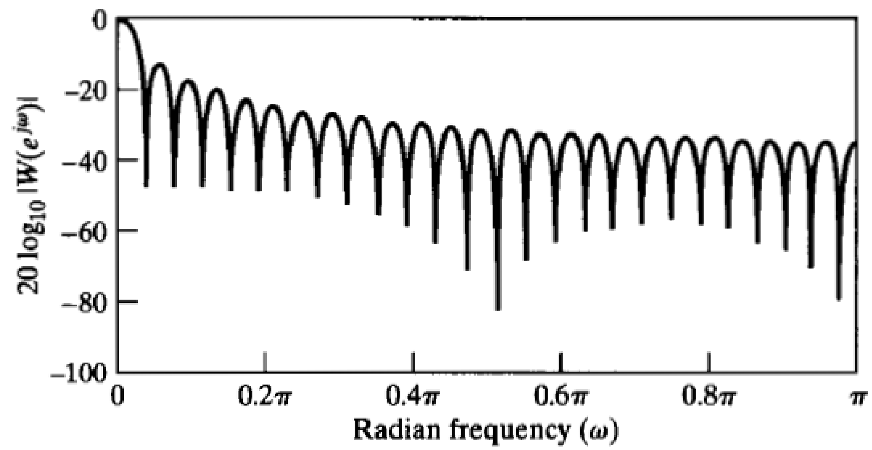
- Application
  - FIR filter design
  - Spectral analysis

- Properties
  - Simple functional form
    - Computation
  - Spectra are concentrated around zero frequency

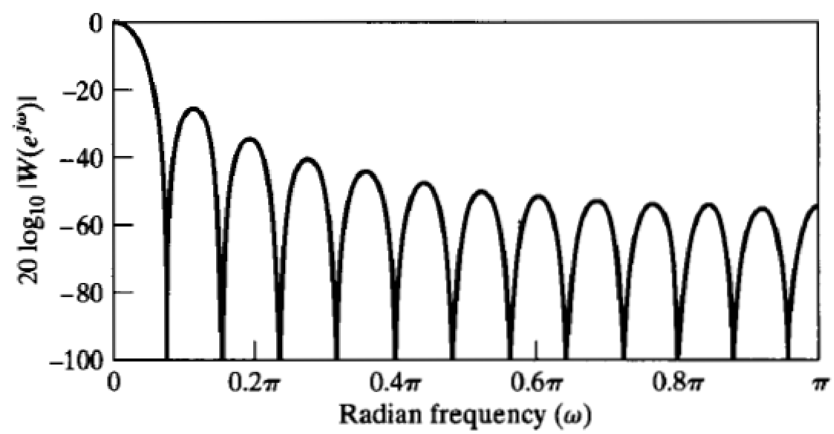
Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Spectra



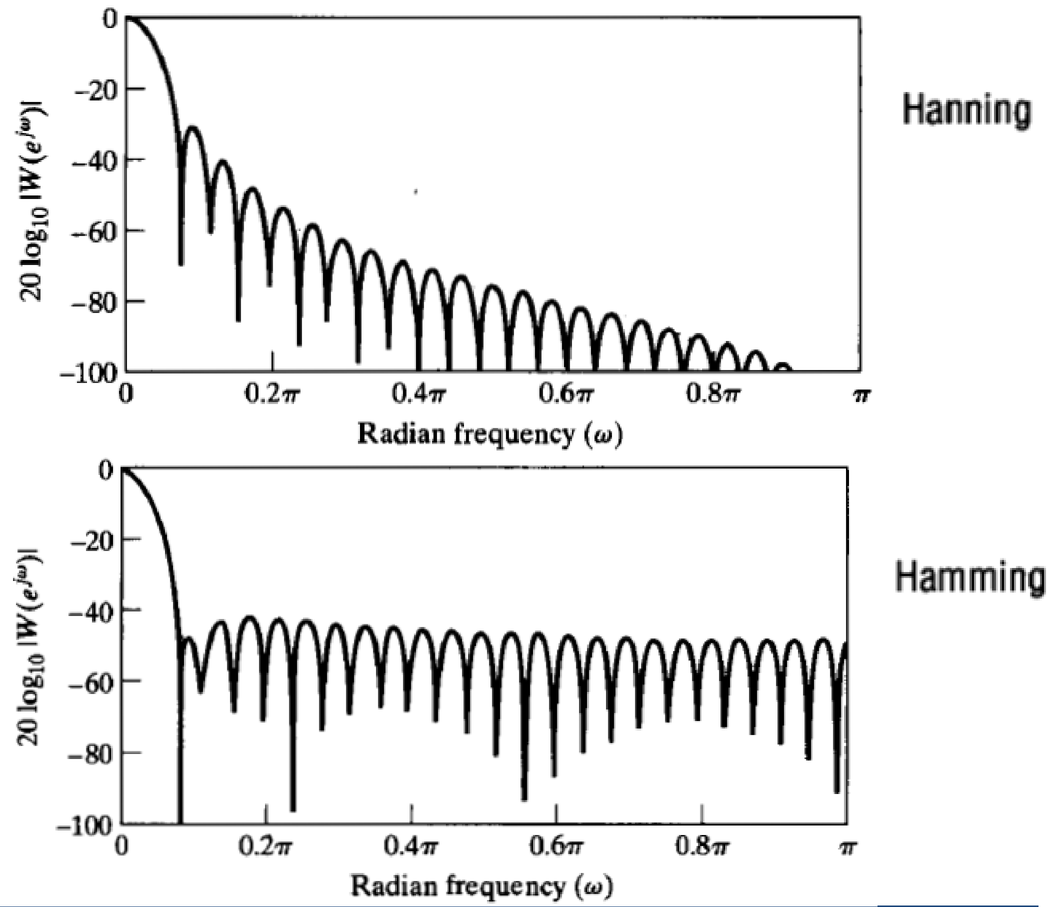
Rectangular.



Bartlett

Filter Design

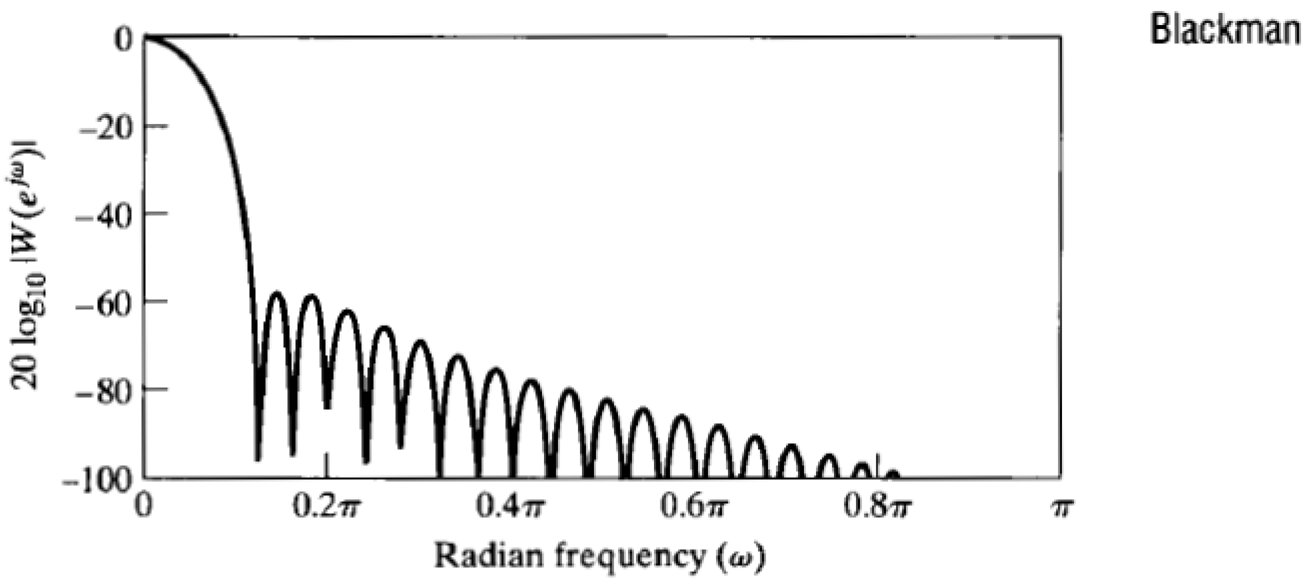
- Spectra



Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Spectra



Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Summary of windows

TABLE COMPARISON

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe
Rectangular	−13	$4\pi/(M+1)$
Bartlett	−25	$8\pi/M$
Hanning	−31	$8\pi/M$
Hamming	−41	$8\pi/M$
Blackman	−57	$12\pi/M$

- Cost of lower sidelobe level: wider mainlobe

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Phase issue
  - Window symmetry around M/2

$$w[n] = \begin{cases} w[M-n], & 0 \le n \le M, \\ 0, & \text{otherwise;} \end{cases}$$

  - and

$$W(e^{j\omega}) = W_e(e^{j\omega})e^{-j\omega M/2}$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

– Symmetric impulse response

$$h_d[M-n] = h_d[n],$$

– Then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2},$$

– Antisymmetric impulse response

$$h_d[M-n] = -h_d[n]$$

$$H(e^{j\omega}) = jA_o(e^{j\omega})e^{-j\omega M/2}$$

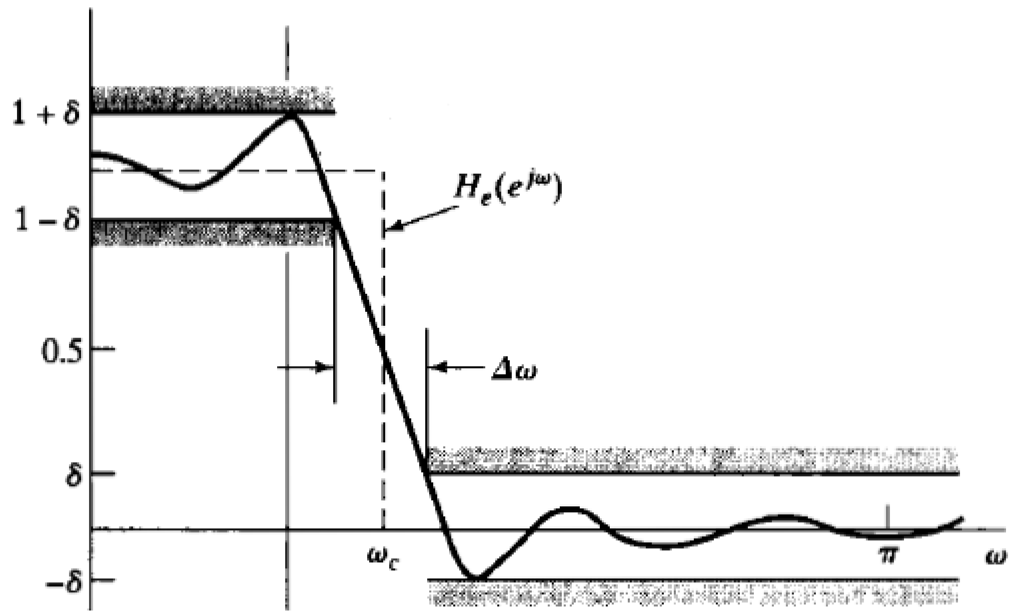
– Even/odd real function

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Example: linear phase LPF

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases}$$



Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

– We obtain:

$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega M/2} e^{j\omega n} d\omega = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} \qquad -\infty < n < \infty.$$

– With the following symmetry property

$$h_{lp}[M - n] = h_{lp}[n]$$

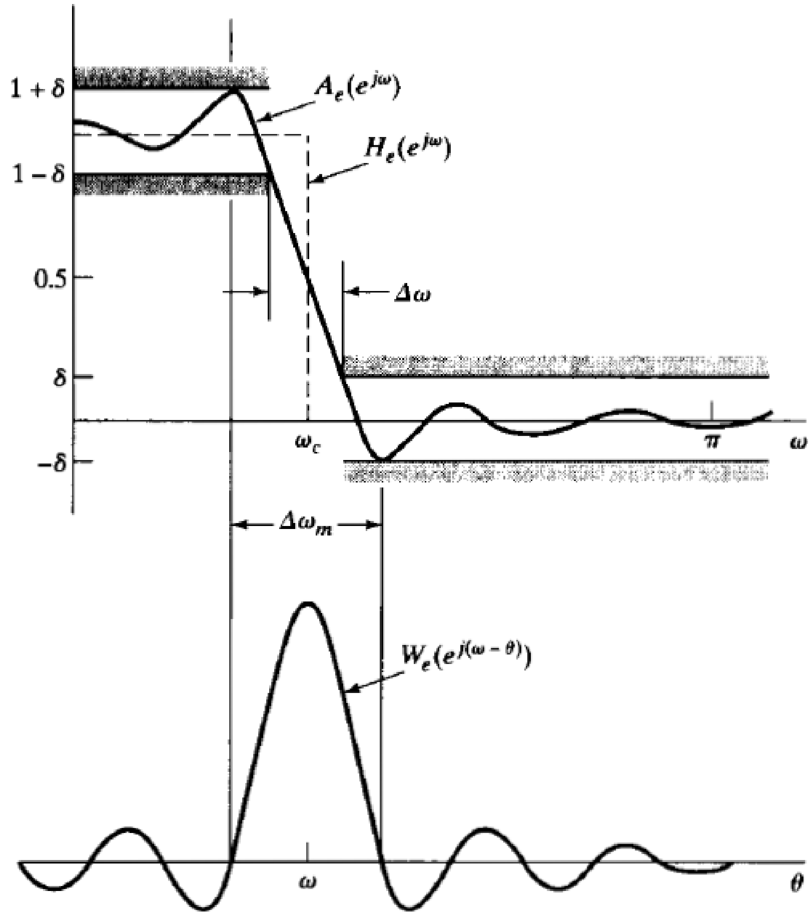
– To obtain a linear phase system

$$h[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} w[n],$$

Filter Design

- Illustration

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$$



Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Comparison

COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hanning	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Wider mainlobe: wider transition time
- Smaller sidelobe level: lower ripple in stopband
- Given filter specification: choosing the proper window and its parameters
  - Try and error!
  - Kaiser method: a systematic way

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Kaiser: a near-optimal window
  - For a given length, having the narrowest mainlobe and the lowest possible sidelob level
  - Maximally concentrated around zero frequency

$$w[n] = \begin{cases} \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha = M/2$, and $I_0(\cdot)$ represents the zeroth-order modified Bessel function

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

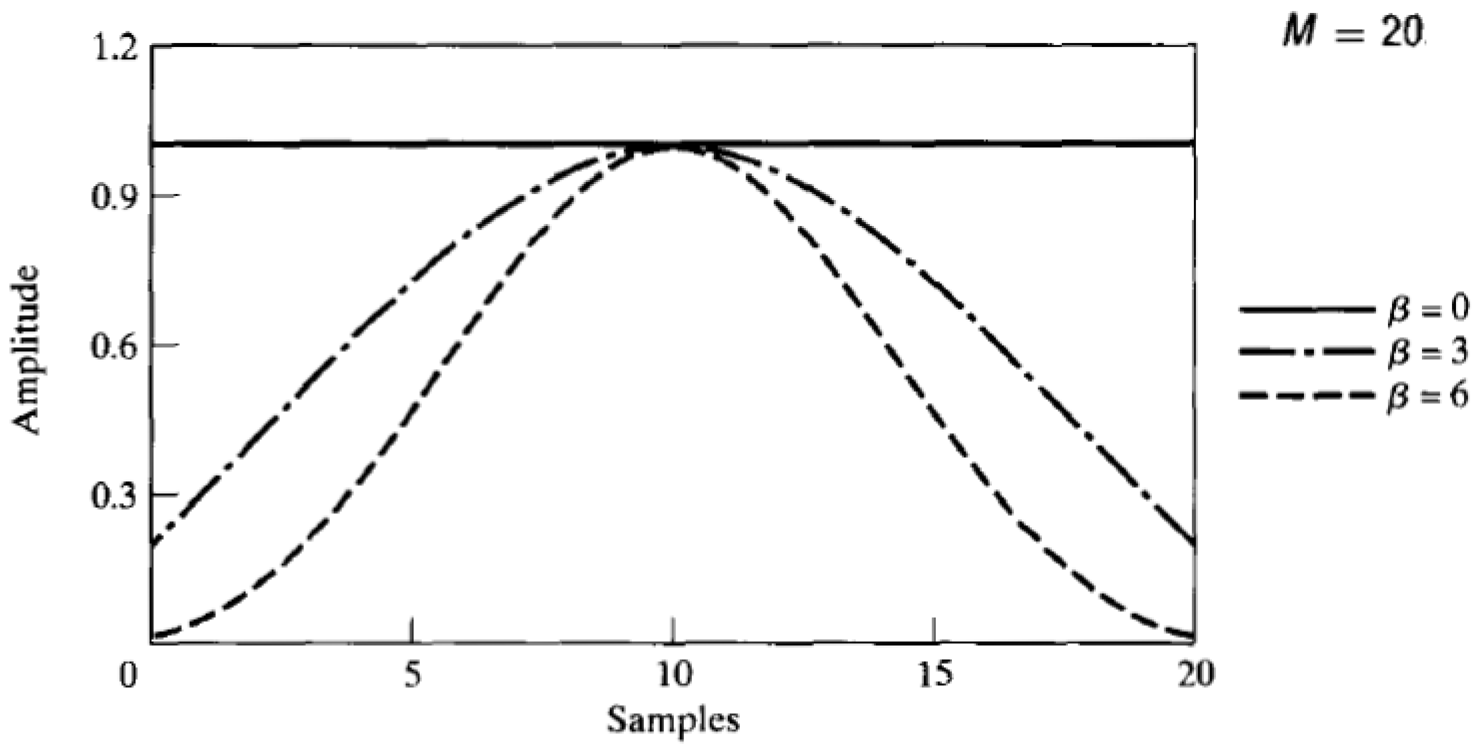
Filter Design

- Parameters:
  - length/shape
  - Trade between sidelobe/mainlobe
  - Higher beta leads to lower sidelobe level
    - Fixed length
  - Higher M leads to narrower mainlobe
    - Fixed beta

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

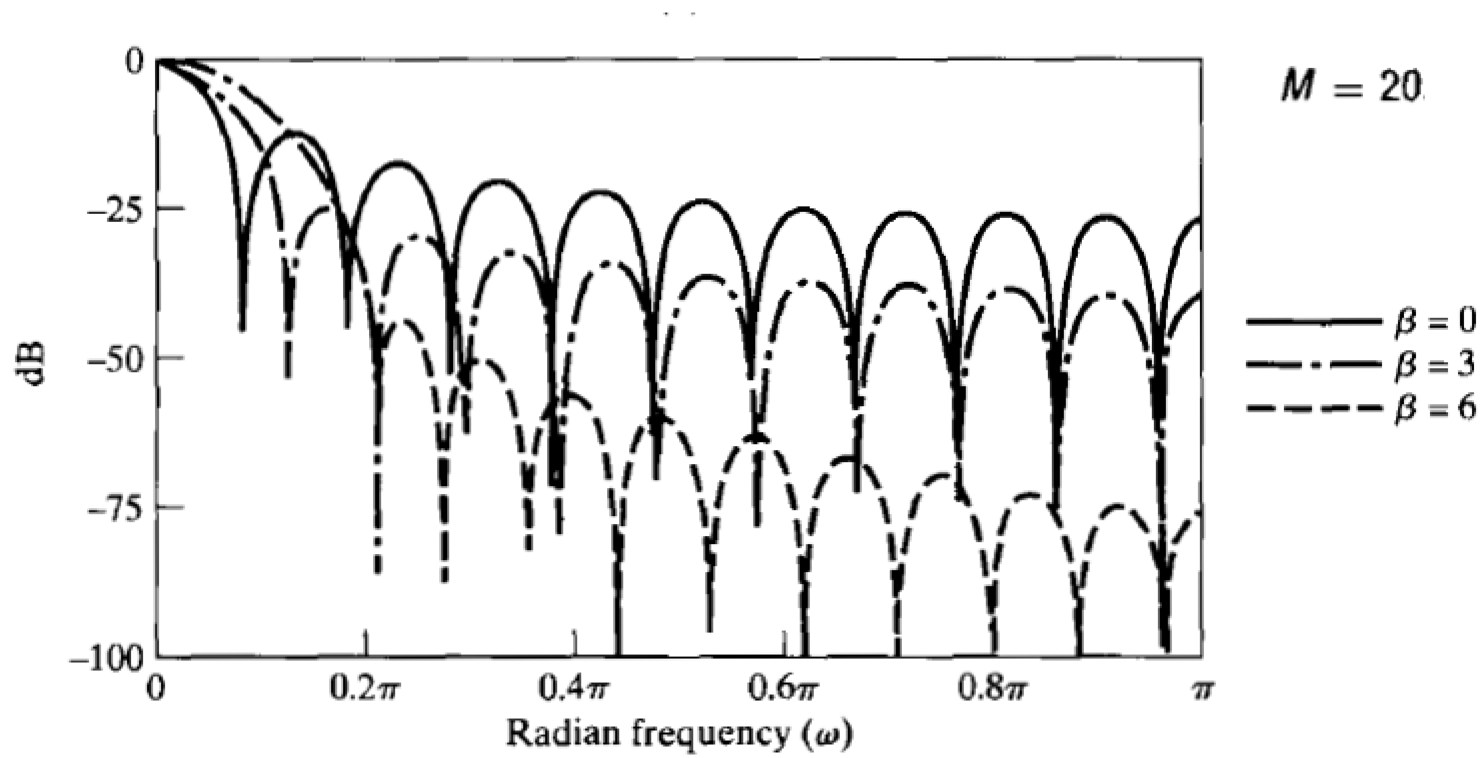
- Time domain



Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

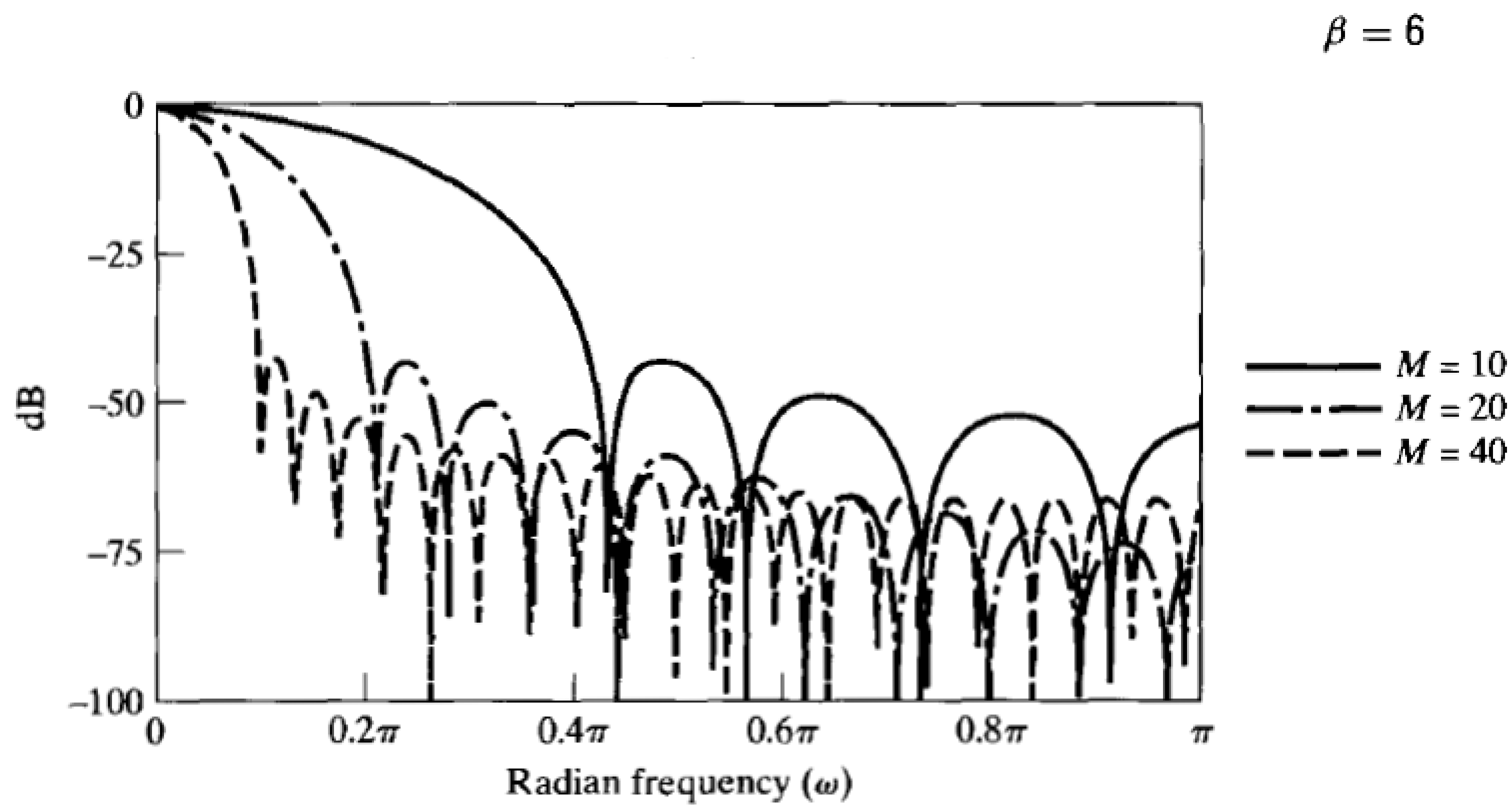
- Frequency



Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Frequency



Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Useful formula to meet a given specification
  - Obtained via extensive numerical examples
  - Passband

$$|H(e^{j\omega})| \geq 1 - \delta$$

  - Stopband

$$|H(e^{j\omega})| \leq \delta$$

  - Transition time

$$\Delta\omega = \omega_s - \omega_p$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

– Define $A = -20\log_{10}\delta$

– Then,

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50, \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50, \\ 0.0, & A < 21. \end{cases}$$

- The zero value: rectangular window

– Smaller delta: larger A which means larger beta

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Also, for the length of the filter

$$M = \frac{A - 8}{2.285\Delta\omega}.$$

  - Approximate value: 2
  - Larger window: narrower mainlobe which means shorter transition time

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Example:
  - Specification

$$\omega_p = 0.4\pi, \omega_s = 0.6\pi, \delta_1 = 0.01, \text{ and } \delta_2 = 0.001$$

  - Note that

$$\omega_c = \frac{\omega_p + \omega_s}{2} = 0.5\pi$$

  - Kaiser parameters

$$\Delta\omega = \omega_s - \omega_p = 0.2\pi, \qquad A = -20\log_{10}\delta = 60.$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

– Using the mentioned expressions:

$$\beta = 5.653, \qquad M = 37$$

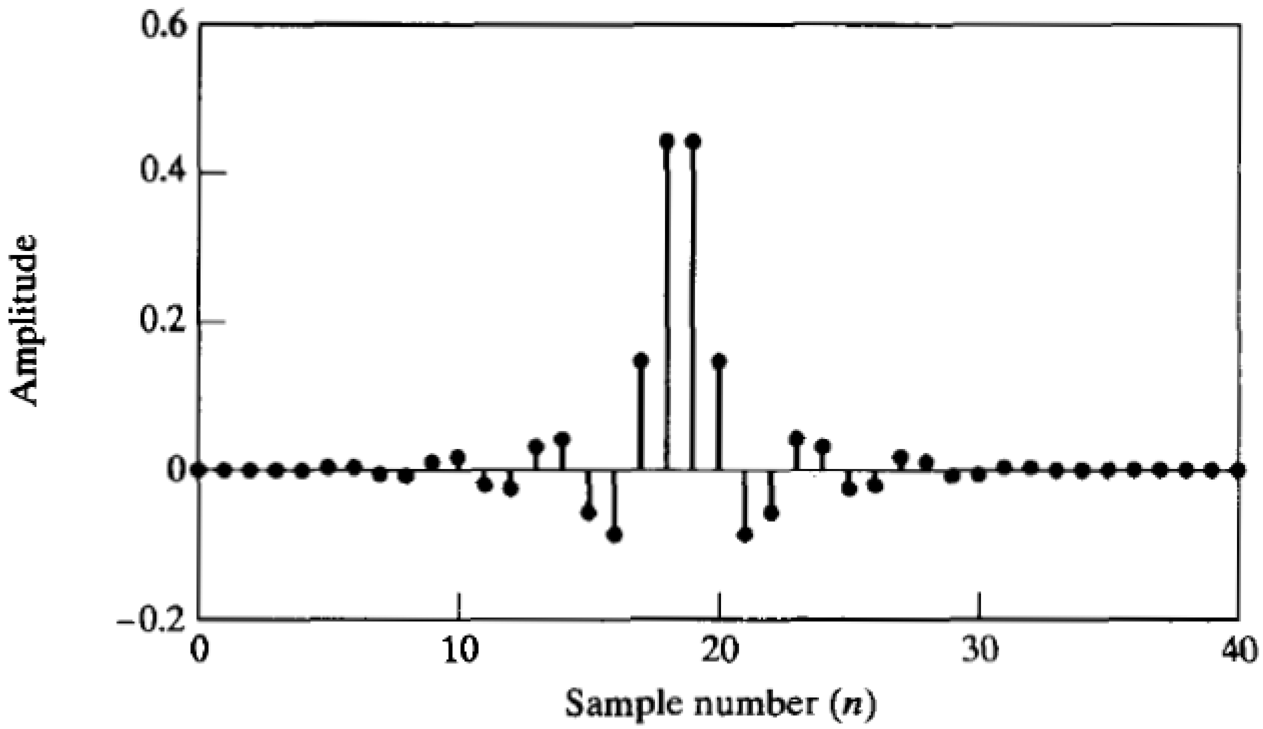
– therefore.,

$$h[n] = \begin{cases} \dfrac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)} \cdot \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M, \\ 0, & \text{otherwise}, \end{cases}$$

$$\alpha = M/2 = 37/2 = 18.5.$$

Filter Design

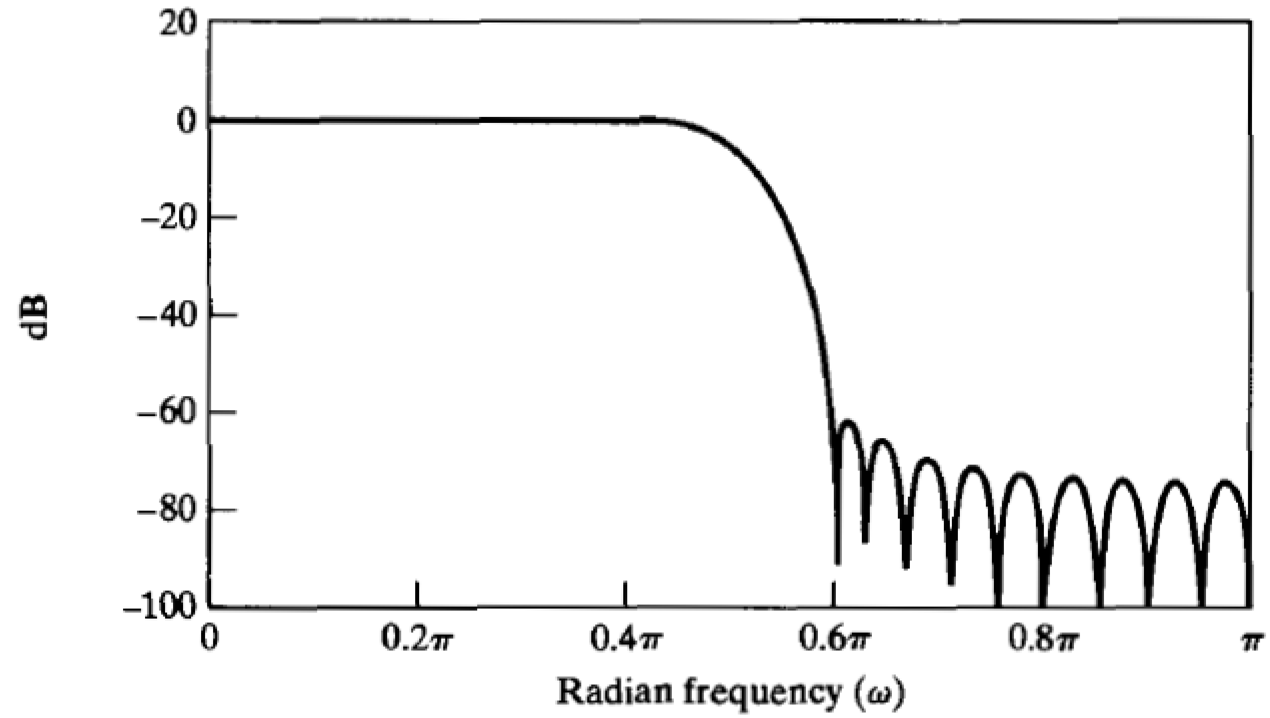
- Impulse response



Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

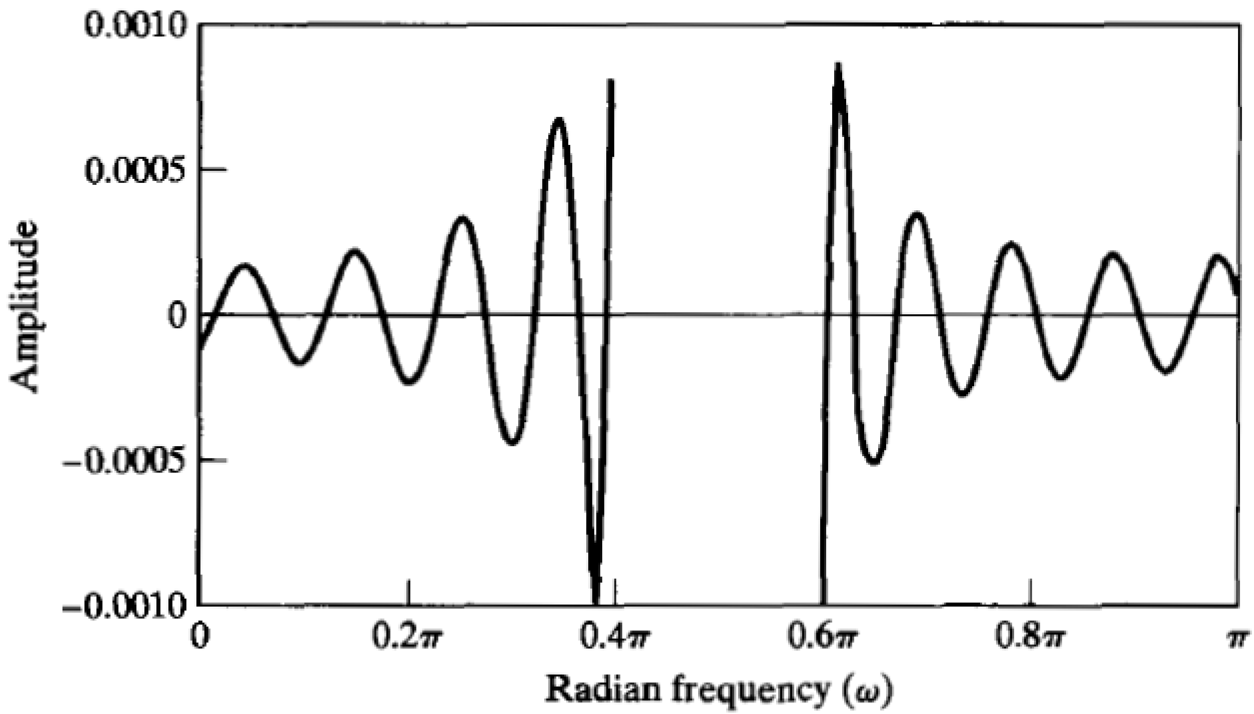
- Designed frequency response



Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Error in frequency domain

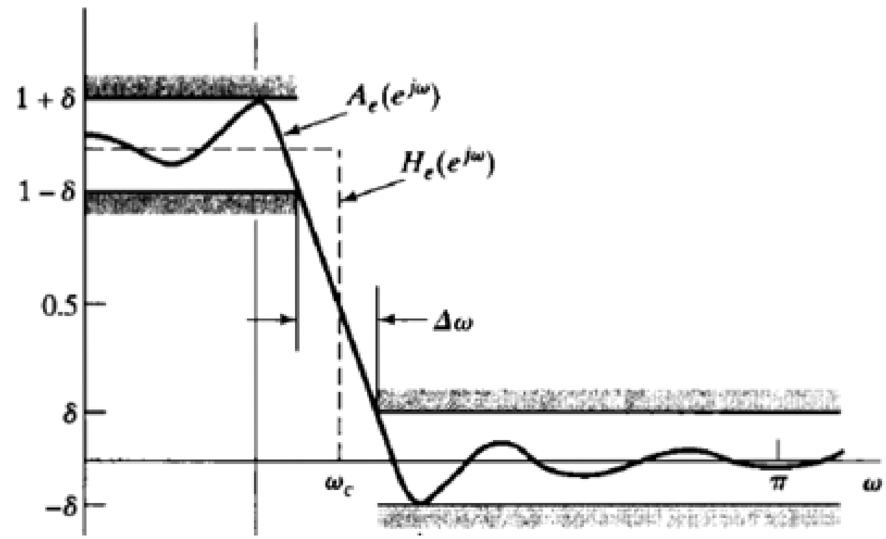


Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- Note that the error id given by

$$E_A(\omega) = \begin{cases} 1 - A_e(e^{j\omega}), & 0 \le \omega \le \omega_p, \\ 0 - A_e(e^{j\omega}), & \omega_s \le \omega \le \pi. \end{cases}$$



  – The phase will be linear!

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Filter Design

- High pass filter

$$H_{\mathrm{hp}}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_c, \\ e^{-j\omega M/2}, & \omega_c < |\omega| \le \pi. \end{cases}$$

  – and

$$H_{\mathrm{hp}}(e^{j\omega}) = e^{-j\omega M/2} - H_{\mathrm{lp}}(e^{j\omega}),$$

  – with

$$h_{\mathrm{hp}}[n] = \frac{\sin \pi(n - M/2)}{\pi(n - M/2)} - \frac{\sin \omega_c(n - M/2)}{\pi(n - M/2)}, \qquad -\infty < n < \infty.$$

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology

Mohammad Mahdi Naghsh
ECE. Dept., Isfahan University of Technology